THE EVOLUTION OF THE PANMICTION POPULATION TAKING INTO ACCOUNT DESTRUCTION OF THE FOETUS

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This article is devoted to the solvability of two models of nonmigrating nonlimited populations proposed in [1]. These models takes into account age and sex of individuals, the panmiction mating and the destruction of the foetus (abortions). One of these models ignored but other one deals with the femals restoration periods after abortions and deliveries. Existence and uniqueness theorems are proved as well as the estimate of solution is obtained.

1. The model ignoring females restoration intervals. We first put in remembrance the following notions used in [1]:

τ_y, τ_x and τ_z are ages of males, females and embryos, respectively, t is time;

$y(t,\tau_y)$, $x(t,\tau_x)$ and $z(t,\tau_y,\tau_x,\tau_z)$ are densities of numbers of males, single and fecundated females, respectively;

$p(t,\tau_y,\tau_x)$ is the females' fecundation rate;

$\nu^y(t,\tau_y)$, $\nu^x(t,\tau_x)$ and $\nu^z(t,\tau_y,\tau_x,\tau_z)$ are death rates of males, single and fecundated females, respectively;

$\chi(t,\tau_y,\tau_x)$ is the abortions rate;

$\sigma_z=(0,\kappa]$, $0<\kappa<\infty$ is the females' gestation interval $\overline{\sigma}_z=[0,\kappa]$, $\sigma_x(\tau_z)=(\tau_{1x}+\tau_z,\ \tau_{2x}+\tau_z]$, $0<\tau_{1x}<\tau_{2x}$;

$\sigma_x(0)$, $\sigma_y=(\tau_{1y},\tau_{2y}]$, where $0<\tau_{1y}<\tau_{2y}<\infty$, and $\sigma_x(\kappa)$ are the females' fecundation interval, the
males' sexual activity interval and the females' reproductivity interval, respectively;

$b^y(t,\tau_y,\tau_x)$ and $b^x(t,\tau_y,\tau_x)$ are the numbers of males and females, born from a female on a delivery;

$[x(t,\tau_i)]$ is a jump of the function x at the line $\tau_x=\tau_i$;

$2^{-1/2}D^y y$, $2^{-1/2}D^x x$ and $3^{-1/2}D^z z$ reprezent directional derivatives along the positive direction of characteristics of operators

$$L^y=\partial/\partial t+\partial/\partial\tau_y,\qquad L^x=\partial/\partial t+\partial/\partial\tau_x,\qquad L^z=L^x+\partial/\partial\tau_z$$

respectively;

$\sigma=\sigma_y\times\sigma_x(\kappa)$, $I=(0,\infty)$, $\bar{I}=[0,\infty)$, $E^y=\{(t,\tau_y)\in I\times I\}$,

$$E^x=\left\{(t,\tau_x)\in I\times\left(I\setminus\bigcup_{i=1}^{4}\tau_i\right),\ \tau_1=\tau_{1x},\ \tau_2=\tau_{1x}+\kappa,\ \tau_3=\tau_{2x},\ \tau_4=\tau_{2x}+\kappa,\ \tau_3-\tau_1>\kappa\right\},$$

$E^z=\{(t,\tau_y,\tau_x,\tau_z)\in I\times\sigma_y\times\sigma_x(\tau_z)\times\sigma_z\}$,

$$\omega(\tau_x)=\sigma_y\times\Omega(\tau_x),\quad \Omega(\tau_x)=\begin{cases}[0,\tau_x-\tau_1], & \tau_x\in(\tau_1,\tau_2],\\ [0,\kappa], & \tau_x\in(\tau_2,\tau_3],\\ [\tau_x-\tau_3,\kappa], & \tau_x\in(\tau_3,\tau_4].\end{cases}$$

The inequality $\tau_3-\tau_1>\kappa$ means the multiple deliveries. In this section the females organism restoration periods after abortions and deliveries will be ignored. The system (see[1])

$$D^y y=-yv^y \quad \text{in } E^y \tag{1}$$

$$D^z z=-zd^z,\ d^z=v^z+\chi \quad \text{in } E^z \tag{2}$$

$$D^x x=-xd^x+X_a+X_b \quad \text{in } E^x \tag{3}$$

$$d^x=v^x+\begin{cases}0, & \tau_x\notin\sigma_x(0),\\ n^{-1}\int\limits_{\sigma_y} yp\,d\tau_y,\quad n=\int\limits_{\sigma_y} y\,d\tau_y, & \tau_x\in\sigma_x(0),\end{cases} \tag{4}$$

$$X_b=\begin{cases}0, & \tau_x\notin\sigma_x(\kappa),\\ \int\limits_{\sigma_y} z|_{\tau_z=\kappa}\,d\tau_y, & \tau_x\in\sigma_x(\kappa).\end{cases} \tag{5}$$

$$X_a=\begin{cases}0, & \tau_x\notin\sigma(\tau_1,\tau_4],\\ \int\limits_{\omega(\tau_x)} z\chi\,d\tau_y d\tau_z, & \tau_x\in(\tau_1,\tau_4]\end{cases} \tag{6}$$

suplemented by the conditions

$$x|_{t=0}=x^0,\quad y|_{t=0}=y^0,\quad z|_{t=0}=z^0 \tag{7}$$

$$y|_{\tau_y=0}=\int\limits_{\sigma} b^y z|_{\tau_z=\kappa}\,d\tau_y d_x,\quad x|_{\tau_x=0}=\int\limits_{\sigma} b^x z|_{\tau_z=\kappa}\,d\tau_y d\tau_x, \tag{8}$$

$$z|_{\tau_z=0}=n^{-1}xyp \tag{9}$$

$$\left[x|_{\tau_x=\tau_i}\right]=0,\quad i=\overline{1,4} \tag{10}$$

governs the evolution of the population. The nonnengative demographic functions v^y, v^x, v^z, p, b^x, b^y, χ and inital functions y^0, x^0, z^0 are assumed to be given. It is also assumed, that x^0, y^0, z^0 satisfy the reconcilable conditions, i.e. conditions (7) – (10) for $t=0$. As it follows from the biological meaning, unknown functions y, x and z must be nonnegative. We note , that (t,τ_x), (t,τ_y) and (t,τ_y,τ_x,τ_z) are the arguments of functions d^x, X_b, X_a, v^y, and d^z, respectively.

Now we consider the solvability problem of the model (1) –(10). Defining

$$a=\sup_{(0,\tau_4]}x^0,\quad p^*=\sup_{\omega_1}p,\quad \chi^*=\sup_{E^z}\chi,\quad d^z_*=\inf_{E^z}d^z,\quad v^x_*=\inf_{E^x}v^x,$$

$$v^y_*=\inf_{E^y}v^y,\quad b^*=\max_{i=x,y}\int\limits_{\sigma_x(k)}\sup_{I\times\sigma_y}b^i d\tau_x,\quad q=\frac{b^*}{a}\int\limits_{\sigma_y}\sup_{\omega_2}z^0 d\tau_y, \tag{11}$$

where $\omega_1=I\times\sigma_y\times\sigma_x(0)$, $\omega_2=\sigma_x(\tau_z)\times\sigma_z$, we formulate the following statement.

THEOREM 1. *Assume that:*

1) $\tau_3-\tau_1>\kappa$,

2) *the nonnegative bounded functions* b^x, b^y *are continuous in* t *and piecewise continuous with respect to* $\tau=(\tau_y,\tau_x)$ *in* $I\times\sigma$,

3) *demographic functions* p, v^x, v^y, v^z, χ *and initial functions* x^0, y^0, z^0, *satisfying the reconcilable-conditions* (7)–(10) for $t=0$, *are continuous and bounded;*

4) *the constans* a, p^*, χ^*, d_*^z, v_*^x, v_*^y, b^*, q, *defined by* (11), *are finite, positive and such that*

$$b^*p^*\exp\left\{-\kappa d_*^z\right\}\le q\le 1,\ q/b^*+\chi^*p^*\kappa\max(1,q/(p^*b^*))\le v_*^x,\ \chi^*p^*\le d_*^z v_*^x\left(1-\exp\left\{-\kappa d_*^z\right\}\right)$$

Then the problem (1)–(10) *has a unique nonnegative solution such that:*

1) y, x *and* z *are continuous functions in* $\bar I\times\bar I$, $\bar I\times\bar I$ *and* $\bar I\times\sigma_y\times\sigma_x(\tau_z)\times\bar{\bar\sigma}_z$, *respectively,*

2) $$\max\left(\sup_t x(t,0)\right),\ \sup_t y(t,0)\le aq^{k+1},\ k\tau_4<t\le(k+1)\tau_4, \tag{12}$$

$$y\le\begin{cases} y^0(\tau_y-t)\exp\left\{-tv_*^y\right\},\ 0<t\le\tau_y<\infty,\\ aq^{k+1}\exp\left\{-\tau_y v_*^y\right\},\ k\tau_4<t-\tau_y\le(k+1)\tau_4,\ \tau_y\in I,\end{cases} \tag{13}$$

$$x\le\begin{cases} x^0(\tau_y-t)\exp\left\{-tv_*^x\right\},\ 0<t\le\tau_x,\ \tau_x\in(0,\tau_2],\\ a,\ 0<t\le\tau_x,\ \tau_x\in(\tau_2,\tau_4],\\ x^0(\tau_x-t)\exp\left\{-tv_*^x\right\},\ 0<t\le\tau_x-\tau_4,\ \tau_x>\tau_4,\\ aq^{k+1},\ k\tau_4<t-\tau_x\le(k+1)\tau_4,\ \tau_x\in(\tau_2,\tau_4],\\ aq^k\exp\left\{-(\tau_x-\tau_4 v_*^x)\right\},\ (k-1)\tau_4<t-\tau_x\le k\tau_4,\ \tau_x>\tau_4,\\ aq^{k+1}\exp\left\{-\tau_x v_*^x\right\},\ k\tau_4<t-\tau_x\le(k+1)\tau_4,\ \tau_x\in(0,\tau_2],\end{cases} \tag{14}$$

$k=0,1,2,\ldots$

Proof. Let's denote: $\tau_0=0$, $\tau_5=\infty$, $d^y=v^y$, $X=X_a+X_b$,

$$\bar x(t)=x|_{\tau_x=0},\ \bar y(t)=y|_{\tau_y=0},\ \bar z(t,\tau_y,\tau_x)=z|_{\tau_z=0},$$

$$F_1(\gamma)\overset{def}{=}\gamma\left(r_0^\gamma\right)\exp\left\{-\int_0^t d^\gamma\left(r_\eta^\gamma\right)d\eta\right\}+\int_0^t\exp\left\{-\int_\alpha^t d^\gamma\left(r_\eta^\gamma\right)d\eta\right\}X\left(r_\alpha^\gamma\right)d\alpha,$$

$$F_2(\gamma,\mu)\overset{def}{=}\gamma\left(h_\mu^\gamma\right)\exp\left\{-\int_\mu^{\tau_\gamma} d^\gamma\left(h_\eta^\gamma\right)d\eta\right\}+\int_\mu^{\tau_\gamma}\exp\left\{-\int_\alpha^{\tau_\gamma} d^\gamma\left(h_\eta^\gamma\right)d\eta\right\}X\left(h_\alpha^\gamma\right)d\alpha, \tag{15}$$

where $X\left(r_\alpha^\gamma\right)=0$, $X\left(h_\alpha^\gamma\right)=0$ for $\gamma=y,\ z$ and $r_\eta^s=(\eta,\eta+\tau_s-t)$, $h_\eta^s=(\eta+t-\tau_s,\eta)$, $s=x,y$, $r_\eta^z=(\eta,\tau_y,\eta+\tau_x-t,\eta+\tau_z-t)$, $h_\eta^z=(\eta+t-\tau_z,\tau_y,\eta+\tau_x-\tau_z,\eta)$. If in (15) the letter γ is not index, then it denotes the respective function. Let $\bar x$, $\bar y$ and $\bar z$ be the continuous functions and assume d^x and X are continuous functions except lines $\tau_x=\tau_i$, $i=\overline{1,4}$. Then from (1)–(3), (7), (15) we obtain the following integral representations

$$y=\begin{cases}F_1(y),\ y(r_0^y)=y^0(\tau_y-t),\ 0\le t\le\tau_y,\\ F_2(y,0),\ y(h_0^y)=\bar y(t-\tau_y),\ 0\le\tau_y<t,\end{cases}\tag{16}$$

$$z=\begin{cases}F_1(z),\ z(r_0^z)=z^0(\tau_y,\tau_x-t,\tau_z-t),\ 0\le t\le\tau_z\le\kappa,\\ F_2(z,0),\ z(h_0^z)=\bar z(t-\tau_z,\tau_y,\tau_x-\tau_z),\ 0\le\tau_z<t,\end{cases}\tag{17}$$

$$x=\begin{cases}F_1(x),\ x(r_0^x)=x^0(\tau_x-t),\ 0\le t\le\tau_x-\tau_i,\ \tau_x\in(\tau_i,\tau_{i+1}],\\ F_2(x,\tau_i),\ t>\tau_x-\tau_i,\ \tau_x\in(\tau_i,\tau_{i+1}],\ x(h_0^x)=\bar x(t-\tau_x),\ t>\tau_x\in[0,\tau_1],\end{cases}\tag{18}$$

where $i=\overline{1,4}$.

The appearance of the index i is conditioned by the jumps of functions d^x, X_a, X_b. Appending to (15)–(18) the formulas (4)–(6), (8), (9) we obtain the system of integral equations.

By using (17) we can write Eqs. (8), (5) and (6) in the form

$$\bar\gamma=\int_\sigma b^\gamma F_1(z)|_{\tau_z=\kappa}d\tau_y d\tau_x,\ \ \gamma=x,y\quad \text{in } [0,\kappa],\tag{19}$$

$$\bar\gamma=\int_\sigma b^\gamma F_2(z,0)|_{\tau_z=\kappa}d\tau_y d\tau_x,\ \ \gamma=x,y\quad \text{in } (\kappa,\infty),\tag{20}$$

$$X_b=\int_{\sigma_y}F_1(z)|_{\tau_z=\kappa}d\tau_y\quad \text{in } [0,\kappa]\times\sigma_x(\kappa),\tag{21}$$

$$X_b=\int_{\sigma_y}F_2(z,0)|_{\tau_z=\kappa}d\tau_y,\quad \text{in } (\kappa,\infty)\times\sigma_x(\kappa)\tag{22}$$

$$X_b=0\quad \text{in } \bar I\times(\bar I\setminus\sigma_x(\kappa)),\tag{23}$$

$$X_a=F_3(z,0,\alpha_1)\ \text{in } \bar I\times(\tau_1,\tau_2],\tag{24}$$

$$X_a=F_3(z,0,\kappa)\ \text{in } \bar I\times(\tau_2,\tau_3],\tag{25}$$

$$X_a=F_3(z,\alpha_2,\kappa)\ \text{in } [\alpha_2,\infty)\times(\tau_3,\tau_4],\tag{26}$$

where $\alpha_1=\tau_x-\tau_1$, $\alpha_2=\tau_x-\tau_3$,

$$F_3(z,\rho,\xi)=\int_{\sigma_y}\left\{\int_\rho^{\beta(\xi)}\chi F_2(z,0)d\tau_z+\int_{\beta(\xi)}^{\xi}\chi F_1(z)d\tau_z\right\}d\tau_y,\ \beta(\xi)=\min(t,\xi).\tag{27}$$

There is the delay argument $t-\kappa$ in Eqs. (19) and (21). Therefore we can consider the problem (15) – (27), (4), (9) going in the consecutive order along the axis t by step κ.

Let $t\in(0,k]$. From Eqs. (15), (19) and (21) it easy to see, that functions $\bar x$, $\bar y$ and X_b are expressed by the values of $F_1(z)|_{\tau_z=\kappa}$ and, consecuently, are known. Thus y given by (16), is also known and the estimates (12), (13) are true. From (15), (9) and (22) we see, that terms, depending on $F_2(z,0)$, in (24)–(26) for $t\le\kappa$ involve the function $x(t-\tau_z,\tau_x-\tau_z)$. For $t\le\kappa$ in Eqs. (24) and (25) $t-\tau_z\le t$, $\tau_x-\tau_z\le\tau_x$, while in Eq. (26) $\tau_x-\tau_z\le\tau_3$. Therefore X_a involves $x(t,\tau_x)$, and (17), where $\tau_x\in(\tau_1,\tau_3]$, is the linear integral equation for x. When $\tau_x\in(0,\tau_1]$, the right–hand side of the Eq. (18) is known, while for $\tau_x>\tau_3$ it involves the only unknown function $x(t,\tau_3)$. When $x(t,\tau_3)$ is obtained the right–band side of (17) is also known. We will obtain $x(t,\tau_3)$ by solving the Eq. (18) for $\tau_x\in(\tau_1,\tau_3]$.

Now we consider the Eq. (18) for $\tau_x \in (\tau_1, \tau_3]$. Let $\Gamma_1 x$ be its right-hand side. By using the conditions of Theorem 1, we can prove, that the operator Γ_1 acts in the class A of nonnegative continuous and bounded by the constant a functions and is contractive. The norm of $x \in A$ is defined as $\sup|x|$. Therefore the equation $x = \Gamma_1 x$ has in A a unique solution. We directly can prove the estimate (14).

Let $t \in (\kappa, 2\kappa]$. By using the known functions x, y and z, when $t \le \kappa$, we obtain for x an integral equation $x = \Gamma_2 x$ when $\tau_x \in (\tau_1, \tau_3]$, and the certain expression when $\tau_x \notin (\tau_1, \tau_3]$. The function y is also known. By using the Banach's principle as above we can prove the unique solvability of equation $x = \Gamma_2 x$ and the estimates (12)–(14).

Going along the axis t and using the same argumentations as above we prove the unique solvability of the problem (15)–(26), (4), (9) and estimates (12)–(14) for $t \in I$. The functions $\bar{x}$, $\bar{y}$, $\bar{z}$, $D^y y$, $D^z z$ are continuous, while $D^x x$ is continuous except the lines $\tau_x = \tau_i$, $i = \overline{1,4}$. This ends the proof of our theorem.

From (12)–(14) one can see, that for $q < 1$ the population vanishes as $t \to \infty$.

NOTE. We considered the case $\tau_3 - \tau_1 > k$. The opposite case can be considered in the same way.

2. **The model taking into account females restoration intervals.** We first briefly define the following notions used in [1]:

τ_y, τ_x and τ_z are ages of males, females and embryos, respectively;

τ_u and τ_v are time passed after an abortion and a delivery, respectively, t is time:

$\sigma_y = (\tau_{1y}, \tau_{2y}]$, $0 < \tau_{1y} < \tau_{2y}$ is the males' sexual activity interval;

$\sigma_z = (0, \kappa_z]$, $0 < \kappa_z < \infty$ is the females' gestation interval, $\bar{\sigma}_z = [0, \kappa_z]$;

$\sigma_u = (0, \kappa_u]$, $0 < \kappa_u < \infty$ and $\sigma_\upsilon = (0, \kappa_\upsilon]$, $0 < \kappa_\upsilon < \infty$ are the females' restoration intervals after abortions and deliveries, respectively, $\bar{\sigma}_u = [0, \kappa_u]$, $\bar{\sigma}_v = [0, \kappa_\upsilon]$;

$\sigma_{xz}(\tau_z) = (\tau_{1x} + \tau_z, \tau_{2x} + \tau_z]$, $0 < \tau_{1x} < \tau_{2x} < \infty$, $\tau \in \bar{\sigma}_z$;

$\sigma_{xz}(0)$ and $\sigma_{xz}(\kappa_z)$ are the females' fecundation and reproductivity intervals;

$\sigma_{xu}(\tau_u) = (\tau_{1x} + \tau_u, \tau_{2x} + \tau_u]$, $\tau_u \in \bar{\sigma}_u$;

$\sigma_{x\upsilon}(\tau_\upsilon) = (\tau_{1x} + \kappa_z + \tau_\upsilon, \tau_{2x} + \kappa_z + \tau_\upsilon]$, $\tau_\upsilon \in \bar{\sigma}_\upsilon$;

$p(t, \tau_y, \tau_x)$ is the females' fecundation rate;

$\nu^y(t, \tau_y)$, $\nu^x(t, \tau_x)$ and $\nu^z(t, \tau_y, \tau_x, \tau_z)$ are death rates of males, single and fecundated females, respectively;

$\nu^u(t, \tau_y, \tau_x, \tau_u)$ and $\nu^\upsilon(t, \tau_y, \tau_x, \tau_\upsilon)$ are death rates of females from restoration intervals after abortions and deliveries, respectively;

$\chi(t, \tau_y, \tau_x, \tau_z)$ is the abortions rate;

$b^y(t, \tau_y, \tau_x)$ and $b^x(t, \tau_y, \tau_x)$ are the numbers of males and females, born from a female on a delivery;

$2^{-1/2} D^y y$, $2^{-1/2} D^x x$, $3^{-1/2} D^z z$, $3^{-1/2} D^u u$ and $3^{-1/2} D^\upsilon \upsilon$ represent directional derivatives along the positive direction of characteristics of operators $L^y = \partial/\partial t + \partial/\partial \tau_y$, $L^x = \partial/\partial t + \partial/\partial \tau_x$, $L^z = L^x + \partial/\partial \tau_z$, $L^u = L^x + \partial/\partial \tau_u$ and $L^\upsilon = L^x + \partial/\partial \tau_\upsilon$, respectively;

$\sigma=\sigma_y\times\sigma_{xz}(\kappa_z)$, $I=(0,\infty)$, $\bar{I}=[0,\infty)$, $E^y=\{(t,\tau_y)\in I\times I\}$, $E^x=\{(t,\tau_x)\in I\times(I\setminus\bigcup_{i=1}^{6}\tau_i)$,

$\tau_1=\tau_{1x},\tau_2=\tau_{1x}+\kappa_u$, $\tau_3=\tau_{1x}+\kappa_z+\kappa_v$, $\tau_4=\tau_{2x}$, $\tau_5=\tau_{2x}+\kappa_z+\kappa_u$, $\tau_6=\tau_{2x}+\kappa_z+\kappa_v$, $\kappa_u<\kappa_\upsilon<\kappa_z, \kappa_u+\kappa_\upsilon+\kappa_z<\tau_{2x}-\tau_{1x}\}$,

$$E^z=\{(t,\tau_y,\tau_x,\tau_z)\in I\times\sigma_y\times\sigma_{xz}(\tau_z)\times\sigma_z\}, E^u=\{(t,\tau_y,\tau_x,\tau_u)\in I\times\sigma_y\times\sigma_{xu}(\tau_u)\times\sigma_u\},$$

$$E^\upsilon=\{(t,\tau_y,\tau_x,\tau_\upsilon)\in I\times\sigma_y\times\sigma_{x\upsilon}(\tau_\upsilon)\times\sigma_\upsilon\};$$

$[x(t,\tau_i)]$ is a jump of the function x at the line $\tau_x=\tau_i$, $i=\overline{1,6}$;

$x^0(\tau_x)$, $y^0(\tau_y)$, $z^0(\tau_y,\tau_x,\tau_z)$, $u^0(\tau_y,\tau_x,\tau_u)$ and $\upsilon^0(\tau_y,\tau_x,\tau_\upsilon)$ are the initial functions;

$$\Omega(\tau_x)=\begin{cases}[0,\tau_x-\tau_1], & \tau_x\in(\tau_1,\tau_1+\kappa_z]\\ [0,\kappa_z], & \tau_x\in(\tau_1+\kappa_z,\tau_4]\\ [\tau_x-\tau_4,\kappa_z], & \tau_x\in(\tau_4,\tau_4+\kappa_z]\end{cases}.$$

The case $\kappa_u<\kappa_\upsilon<\kappa_z$ biologically is frequent and the inequality $\kappa_u+\kappa_\upsilon+\kappa_z<\tau_4-\tau_1$ means the multiple deliveries. The system (see [1])

$$D^y y=-yv^y \text{ in } E^y, \tag{28}$$

$$D^z z=-zd^z, d^z=v^z+\chi, \text{ in } E^z, \tag{29}$$

$$D^u u=-uv^u \text{ in } E^u, \tag{30}$$

$$D^\upsilon \upsilon=-\upsilon v^\upsilon \text{ in } E^v, \tag{31}$$

$$D^x x=-xd^x+X_a+X_b \text{ in } E^x, \tag{32}$$

$$d^x=v^x+\begin{cases}0, & \tau_x\notin\sigma_{xz}(0),\\ n^{-1}\int_{\sigma_y} ypd\tau_y,\ n=\int_{\sigma_y} yd\tau_y, & \tau_x\in\sigma_{xz}(0)\end{cases} \tag{33}$$

$$X_a=\begin{cases}0, & \tau_x\notin\sigma_{xu}(\kappa_u),\\ \int_{\sigma_y} u|_{\tau_u=\kappa_u}d\tau_y, & \tau_x\in\sigma_{xu}(\kappa_u)\end{cases} \tag{34}$$

$$X_b=\begin{cases}0, & \tau_x\notin\sigma_{x\upsilon}(k_\upsilon),\\ \int_{\sigma_y} v|_{\tau_\upsilon=k_\upsilon}d\tau_y, & \tau_x\in\sigma_{x\upsilon}(\kappa_\upsilon)\end{cases} \tag{35}$$

suplemented by the conditions

$$y|_{t=0}=y^0,\ z|_{t=0}=z^0,\ u|_{t=0}=u^0,\ v|_{t=0}=v^0,\ x|_{t=0}=x^0, \tag{36}$$

$$y|_{\tau_y=0}=\int_\sigma b^y z|_{\tau_z=\kappa_z}d\tau_y d\tau_x,\ x|_{\tau_x=0}=\int_\sigma b^x z|_{\tau_z=\kappa_z}d\tau_y d\tau_x, \tag{37}$$

$$\upsilon|_{\tau_\upsilon=0}=z|_{\tau_z=\kappa_z},\ z|_{\tau_z=0}=n^{-1}xyp,\ u|_{\tau_u=0}=\int_{\Omega(\tau_x)}\chi zd\tau_z, \tag{38}$$

$$[x|_{\tau_x=\tau_i}]=0,\ i=\overline{1,6} \tag{39}$$

governs the evolution of the population. The nonnegative demographic functions ν^y, ν^x, ν^z, ν^u, ν^υ, p, b^x, b^y, χ and initial functions y^0, x^0, z^0, u^0, υ^0 are assumed to be given. It is assumed also, that inital functions satisfy the reconcilable conditions, i.e. conditions (37)–(39) for $t=0$. As it follows from the biological meaning, unknown functions x, y, z, υ must be nonnegative.

We note, that (t,τ_x) and (t,τ_y,τ_x,τ_z) are the arguments of functions d^x, X_a, X_b and d^z, respectively.

Now we will consider the solvability problem of the model (28)–(39). Defining

$$a=\sup_{[0,\tau_6]} x^0,\ p^*=\sup_{\omega_0} p,\ \chi^*=\sup_{E^z}\chi,\ d_*^z=\inf_{E^z} d^z,\ \nu_*^x=\inf_{E^x}\nu^x,\ \nu_*^y=\inf_{E^y}\nu^y,$$

$$b^*=\max_{i=x,y}\int_{\sigma_{xz}(k_z)}\sup_{I\times\sigma_y} b^i d\tau_x,\quad q=\frac{b^*}{a}\max\Big(\int_{\sigma_y}\sup_{\omega_1} z^0 d\tau_y,\ \int_{\sigma_y}\sup_{\omega_2}\upsilon^0 d\tau_y\Big), \tag{40}$$

$\omega_0=\bar{I}\times\sigma_y\times\sigma_{xz}(0)$, $\omega_1=\sigma_{xz}(\tau_z)\times\sigma_z$, $\omega_2=\sigma_{x\upsilon}(\tau_\upsilon)\times\sigma_\upsilon$, $\omega_3=\sigma_{xu}(\tau_u)\times\sigma_u$, $\tilde{\tau}_{2x}=\tau_{2x}+\kappa_z$ we can formulate the following statement.

THEOREM 2. *Assume that:*

1) $\kappa_u<\kappa_\upsilon<\kappa_z$, $\tau_4-\tau_1>\kappa_u+\kappa_\upsilon+\kappa_z$;

2) *the nonnegative bounded functions* b^x, b^y *are continuous in t and piecewise continuous with respect to* $\tau=(\tau_y,\tau_x)$ in $I\times\sigma$;

3) *demographic functions* p, χ, ν^x, ν^y, ν^z, ν^u, ν^υ *and inital functions* x^0, y^0, z^0, u^0, υ^0 *satisfying the reconcilable conditions* (37)–(39) *for* $t=0$, *are continuous and bounded;*

4) *the constants* a, p^*, χ^*, d_*^z, ν_*^x, ν_*^y, b^*, q, *defined by* (40), *are finite, positive and such that:*

$$\int_{\sigma_y}\sup_{\omega_3} u^0 d\tau_y\le\chi^* p^*\kappa_z\max(1,q/p^*b^*),\quad b^*p^*\exp\{-\kappa_z d_*^z\}\le q\le 1,$$

$$q/b^*+\chi^*p^*\kappa_z\max(1,q/p^*b^*)\le\nu_*^x.$$

Then the problem (28)–(30) *has a unique nonnegative solution such that:*

1) y, x, z, u, υ *are continuous functions in the domains* $\bar{I}\times\bar{I}$, $\bar{I}\times\bar{I}$, $\bar{I}\times\sigma_y\times\sigma_{xz}(\tau_z)\times\bar{\sigma}_z$, $\bar{I}\times\sigma_y\times\sigma_{xu}(\tau_u)\times\bar{\sigma}_u$, *and* $\bar{I}\times\sigma_y\times\sigma_{x\upsilon}(\tau_\upsilon)\times\bar{\sigma}_\upsilon$, *respectively;*

2) $$\max(\sup x(t,0)),\ \sup y(t,0)\le aq^{k+1},\ k\tilde{\tau}_{2x}<t\le(k+1)\tilde{\tau}_{2x}, \tag{41}$$

$$y\le\begin{cases} y^0(\tau_y-t)\exp\{-t\nu_*^y\}, & 0\le t\le\tau_y<\infty,\\ aq^{k+1}\exp\{-\tau_y\nu_*^y\}, & k\tilde{\tau}_{2x}<t-\tau_y\le(k+1)\tilde{\tau}_{2x},\ \tau_y\in I,\end{cases} \tag{42}$$

$$x \le \begin{cases} x^0(\tau_x - t)\exp\{-t\nu_*^x\},\ 0<t\le\tau_x,\ \tau_x\in(0,\tau_2], \\ a,\ 0<t\le\tau_x,\ \tau_x\in(\tau_2,\tau_6], \\ x^0(\tau_x - t)\exp\{-t_x\nu_*^x\},\ 0<t\le\tau_x-\tau_6,\ \tau_x>\tau_6, \\ aq^{k+1}\exp\{-\tau_x\nu_*^x\},\ k\tilde{\tau}_{2x}<t-\tau_x\le(k+1)\tilde{\tau}_{2x},\ \tau_x\in(0,\tau_2], \\ aq^{k+1},\ k\tilde{\tau}_{2x}<t-\tau_x\le(k+1)\tilde{\tau}_{2x},\ \tau_x\in(\tau_2,\tau_6], \\ aq^k\exp\{-(\tau_x-\tau_6)\nu_*^x\},\ (k-1)\tilde{\tau}_{2x}<t-\tau_x\le k\tilde{\tau}_{2x},\ \tau_x>\tau_6, \\ a\exp\{-(\tau_x-\tau_6)\nu_*^x\},\ \tau_x-\tau_6<t\le\tau_x-\tilde{\tau}_{2x},\ \tau_x>\tau_6, \end{cases} \tag{43}$$

$k=0,1,2,...$

Proof. The problem (28)−(39) can be solved by the method used in section 1. Assume $\tau_0=0$, $\tau_7=\infty$.

By using the conditions of our theorem and the functions $F_1(\gamma)$, $F_2(\gamma,\mu)$, defined by (15), where $X(r_\alpha^\gamma)=X(h_\alpha^\gamma)=0$ for $\gamma\neq x$, we can write the solution of the problem (28)−(39) in the integral form:

$$y=\begin{cases} F_1(y),\ y(r_0^y)=y^0(\tau_y-t),\ 0\le t\le\tau_y, \\ F_2(y,0),\ y(h_0^y)=\bar{y}(t-\tau_y),\ 0\le\tau_y<t, \end{cases} \tag{44}$$

$$z=\begin{cases} F_1(z),\ z(r_0^z)=z^0(\tau_y,\tau_x-t,\tau_z-t),\ 0\le t\le\tau_z\le\kappa_z, \\ F_2(z,0),\ z(h_0^z)=\bar{z}(t-\tau_z,\tau_y,\tau_x-\tau_z),\ 0\le\tau_z<t, \end{cases} \tag{45}$$

$$x=\begin{cases} F_1(x),\ x(r_0^x)=x^0(\tau_x-t),\ 0\le t\le\tau_x-\tau_i,\ \tau_x\in(\tau_i,\tau_{i+1}], \\ F_2(x,\tau_i),\ t>\tau_x-\tau_i,\ \tau_x\in(\tau_i,\tau_{i+1}],\ x(h_0^x)=\bar{x}(t-\tau_x),\ t>\tau_x\in[0,\tau_1] \end{cases} \tag{46}$$

for $i=\overline{0,6}$,

$$u=\begin{cases} F_1(u),\ u(r_0^u)=u^0(\tau_y,\tau_x-t,\tau_u-t),\ 0\le t\le\tau_u\le\kappa_u, \\ F_2(u,0),\ u(h_0^u)=\bar{u}(t-\tau_u,\tau_y,\tau_x-\tau_u),\ 0\le\tau_u<t, \end{cases} \tag{47}$$

$$\upsilon=\begin{cases} F_1(\upsilon),\ \upsilon(r_0^\upsilon)=\upsilon^0(\tau_y,\tau_x-t,\tau_\upsilon-t),\ 0\le t\le\tau_\upsilon\le\kappa_\upsilon, \\ F_2(\upsilon,0),\ \upsilon(h_0^\upsilon)=\bar{\upsilon}(t-\tau_\upsilon,\tau_y,\tau_x-\tau_\upsilon),\ 0\le\tau_v<t, \end{cases} \tag{48}$$

where $r_\eta^s=(\eta,\eta+\tau_s-t)$, $h_\eta^s=(\eta+t-\tau_s,\eta)$ for $s=x,y$ and $r_\eta^s=(\eta,\tau_y,\eta+\tau_x-t,\eta+\tau_s-t)$, $h_\eta^s=(\eta+t-\tau_s,\tau_y,\eta+\tau_x-\tau_s,\eta)$ for $s=z,u,\upsilon$ while

$$\bar{\upsilon}(t,\tau_y,\tau_x)=\bar{z}|_{\tau_z=\kappa_z},\ \bar{u}(t,\tau_y,\tau_x)=\int\limits_{\Omega(\tau_x)}\chi z d\tau_z. \tag{49}$$

Consider the functions $X_b, \bar{x}, \bar{y}, X_a$ in detail. Formulas (45),(48) show, that X_b is known for $(t,\tau_x)\in(0,\kappa_\upsilon+\kappa_z]\times(\tau_3,\tau_6]$, while for $(t,\tau_x)\in(\kappa_\upsilon+\kappa_z,\infty)\times(\tau_3,\tau_6]$ it is represented by formula

$$X_b(t,\tau_x)=\int_{\sigma_y}\left(n^{-1}xy\right)_{(\xi_1,\tau_y,\eta_1)}f_1(t,\tau_y,\tau_x)d\tau_y, \tag{50}$$

where $\xi_1=t-\kappa_\upsilon-\kappa_z$, $\eta_1=\tau_x-\kappa_\upsilon-\kappa_z$, and f_1 is known. The functions $\bar{x},\bar{y}$ are represented by (19),(20) where κ must be replaced by κ_z. Eqs. (47) and (50) show, that X_a is known for $(t,\tau_x)\in(0,\kappa_u]\times(\tau_2,\tau_5]$ and

$$X_a(t,\tau_x)=\int_{\sigma_y}f_2(\xi_2,\tau_y,\eta_2)\left(\int_{\Omega(\eta_2)}(\chi z)_{(\xi_2,\tau_y,\eta_2,\tau_z)}d\tau_z\right)d\tau_y \tag{51}$$

for $(t,\tau_x)\in(\kappa_u,\infty)\times(\tau_2,\tau_5]$, where $\xi_2=t-\kappa_u$, $\eta_2=\tau_x-\kappa_u$, and f_2 is known.

Using (45) from (20) with κ replaced by κ_z and (51) we can eliminate z and, cosecuently, obtain for y and x the equations with delay arguments. We can solve these equations going along the axis t with the step $\rho=\min(\kappa_u,\tau_1)$, because x,y for $t\in[0,\rho]$ are known.

The continuity of the solution, mentioned in our theorem, and its differentiability along the respective characteristics are ensured by the conditions of theorem. The estimates (41)–(43) may be obtained directly by using the Gronwall's inequality. The estimate (43) shows, that for $q<1$ the population vanishes as $t\to\infty$. This ends the proof.

REFERENCES

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