Mathematical Modelling and Analysis
Volume 25, Issue 4, 661–679, 2020
https://doi.org/10.3846/mma.2020.10517

http://mma.vgtu.lt
ISSN: 1392-6292
eISSN: 1648-3510

An Overlapping Schwarz Method for Singularly Perturbed Fourth-Order Convection-Diffusion Type

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Received June 13, 2019; revised August 31, 2020; accepted August 31, 2020

Abstract. In this paper, we have constructed an iterative numerical method based on an overlapping Schwarz procedure with uniform mesh for singularly perturbed fourth-order of convection-diffusion type. The method splits the original domain into two overlapping subdomains. A hybrid difference scheme is proposed in which on the boundary layer region we use the central finite difference scheme on a uniform mesh while on the non-layer region we use the mid-point difference scheme on a uniform mesh. It is shown that the method produces numerical approximations which converge in the maximum norm to the exact solution. We prove that, when appropriate subdomains are used the method produces convergence of almost second-order. Furthermore, it is shown that, two iterations are sufficient to achieve the expected accuracy. Numerical examples are presented to support the theoretical results.

Keywords: singularly perturbed problems, convection-diffusion equations, Schwarz method, hybrid difference scheme.

AMS Subject Classification: 65L10.

1 Introduction

Singular Perturbation Problems (SPPs) appear in many branches of applied mathematics, like fluid dynamics, quantum mechanics, turbulent interaction of waves and currents, electrodes theory, etc. The convergence of the numerical approximations generated by standard numerical methods applied to such problems depends adversely on the singular perturbation parameter. Most of these works have concentrated on second-order single differential equations ([4] and the references therein), but for fourth-order equations only few results are reported in the literature [2, 15, 16, 17].

Numerical methods for singularly perturbed problems comprising domain decomposition and Schwarz iterative techniques have been examined by various authors, for example, in [1, 6, 7, 8, 9, 10, 18, 20]. In [10], the authors examined a continuous overlapping Schwarz method for a singularly perturbed convection-diffusion equation with arbitrary fixed interface positions and found it to be uniformly convergent with respect to the perturbation parameter. In [20], an analysis of overlapping domain decomposition methods for singularly perturbed reaction-diffusion problems with distinct small positive parameters is presented. The authors of [20] found a flaw in the analysis of domain decomposition methods explored in [6, 13, 18]. The authors observation is that the constant C is not independent of the iteration number k and it is growing at each induction step in their proof of [6, 13, 18]. But in [20] the authors have presented an alternate analysis of overlapping domain decomposition methods for singularly perturbed reaction-diffusion problems with two parameters and problems in [18].

The authours of [8,9] have concluded that the numerical solution of classical finite difference scheme used in Schwarz method does not converge to the exact solution of their test problem which is a single equation. But our proposed scheme used in Schwarz method [3] has overcome the fundamental difficulty mentioned by the authours of [8,9]. In [8,9], the authors used the same scheme in both the layer and non-layer regions, whereas in our case we used different schemes in each region.

As far as the authors knowledge goes fourth-order SPPs have not yet been examined for higher-order of convergence. Therefore, we are interested in constructing a numerical method for fourth-order SPPs. Of primary interest we have been proved that when appropriate subdomains are used the method produce convergence of almost second-order.

Motivated by the works of [2,10,15,16,17] we have examined experimentally the performance of Schwarz method to the fourth-order Singularly Perturbed Boundary Value Problems (SPBVPs) described as below.

$$-\varepsilon y^{iv}(x) + a(x)y'''(x) + b(x)y''(x) - c(x)y(x) = -f(x), \quad x \in \Omega, \tag{1.1}$$

$$y(0) = q_1, \quad y''(0) = -q_2, \quad y(1) = q_3 \quad y''(1) = -q_4, \tag{1.2}$$

where $a(x), b(x),\ c(x)$ are sufficiently smooth functions satisfying the following conditions:

$$a(x) \geq \alpha, \quad \alpha > 1, \tag{1.3}$$

$$b(x) \geq 0,$$

$$0 \geq c(x) \geq -\gamma, \quad \gamma > 0, \tag{1.4}$$

$$\alpha > 5\gamma \tag{1.5}$$

with $0 < \varepsilon \ll 1$, $\Omega = (0,1)$, $\bar{\Omega} = [0,1]$ and $y \in C^{(4)}(\Omega) \cap C^{(2)}(\bar{\Omega})$, which have important applications in fluid dynamics, have been studied in [5], and the references therein. The SPBVPs (1.1)–(1.2) can be transformed into an equivalent weakly coupled system of two ODEs subject to suitable boundary conditions of the form:

$$\begin{cases} L_1\boldsymbol{y}(x) \equiv -y_1''(x) - y_2(x) = 0, \quad x \in \Omega, \\ L_2\boldsymbol{y}(x) \equiv -\varepsilon y_2''(x) + a(x)y_2'(x) \\ \qquad\qquad +b(x)y_2(x) + c(x)y_1(x) = f(x), \quad x \in \Omega, \end{cases} \tag{1.6}$$

$$y_1(0) = q_1, \quad y_2(0) = q_2, \quad y_1(1) = q_3, \quad y_2(1) = q_4, \tag{1.7}$$

where $\boldsymbol{y} = (y_1, y_2)^T$ and $a(x),\ b(x), c(x)$ are sufficiently smooth functions satisfying (1.3)–(1.5). The above weakly coupled system can be written in the matrix-vector form as

$$\mathbf{L}\boldsymbol{y} \equiv \begin{pmatrix} L_1\boldsymbol{y} \\ L_2\boldsymbol{y} \end{pmatrix} \equiv \begin{pmatrix} -\frac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon\frac{d^2}{dx^2} \end{pmatrix} \boldsymbol{y} + \mathbf{A}(x)\boldsymbol{y}' + \mathbf{B}(x)\boldsymbol{y} = \mathbf{f}(x),\ x \in \Omega,$$

$$\boldsymbol{y}(0) = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad \boldsymbol{y}(1) = \begin{pmatrix} q_3 \\ q_4 \end{pmatrix},$$

where $\boldsymbol{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $\mathbf{f}(x) = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$, $\mathbf{A}(x) = \begin{pmatrix} 0 & 0 \\ 0 & a(x) \end{pmatrix}$ and $\mathbf{B}(x) = \begin{pmatrix} 0 & -1 \\ c(x) & b(x) \end{pmatrix}$. Let $\beta = \min\{-1, b(x) + c(x)\}$.

In this paper, of primary interest we have proved that discrete Schwarz method converge to the solution of the continuous problem. The method is shown to be of almost second-order convergence. Furthermore, we show that, just two iterations are required to achieve the expected accuracy.

Remark 1. The solution of the problem (1.1)–(1.2) exhibits a boundary layer at $x = 1$ which is less severe because the boundary conditions are prescribed for the derivative of the solution [14]. The condition (1.3) says that (1.1)–(1.2) is a non-turning point problem. The condition (1.4) is known as the quasi-monotonicity condition [14]. The maximum principle theorem for the above system (1.1)–(1.2) and for the corresponding discrete problem are established using the conditions (1.3)–(1.4) and using this principle, we can establish a stability result.

The outline of rest of the paper is as follows. In Section 2, the continuous Schwarz method is described. The derivative estimates are obtained in Section 3. In Section 4, the discrete Schwarz method is described. The maximum pointwise error bounds are obtained in Section 5. Numerical experiments are presented in Section 6 and finally, conclusions are included in Section 7.

Notations: Throughout the paper we use C, with or without subscript to denote a generic positive constant independent of ε, the iteration k and the discretization parameter N.

Let $\boldsymbol{y} : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$. The appropriate norm for studying the convergence of the numerical solution to the exact solution of a SPP is $\|y\|_D = \sup_{x\in D} |y(x)|$. For a vector $\boldsymbol{y} = (y_1, y_2)^T$, we define $\|\boldsymbol{y}\| = \max_{j=1,2} |y_j|$.

For a vector valued function $\boldsymbol{z} = (z_1, z_2)^T$, define $\|\boldsymbol{z}\|_{\Omega} = \max\{\|z_1\|_{\Omega}, \|z_2\|_{\Omega}\}$. Given any two vector valued functions, $\boldsymbol{z}$ and $\boldsymbol{y}$, $\boldsymbol{z} \geq \boldsymbol{y}$ if $z_j \geq y_j$ for all $j = 1, 2$. For a vector of mesh functions $\mathbf{Z}(x_i) = (Z_1(x_i), Z_2(x_i))^T$, define

$$\|\boldsymbol{Z}\|_{\Omega^N} = \max_{j=1,2} \left(\max_{x_i \in \Omega^N} |Z_j(x_i)| \right).$$

2 Continuous Schwarz method

In this section, a continuous Schwarz method is described. This process generates a sequence of iterates $\{\boldsymbol{y}^{[k]}\}$, which converges as $k \to \infty$ to the exact solution $\boldsymbol{y}$. Further we prove the maximum principle for (1.6)–(1.7). Using this principle, a stability result is stated. First, we split the domain into two overlapping subdomains as $\Omega_c = (0, 1-\tau)$ and $\Omega_r = (1-2\tau, 1)$, where the subdomain transition parameter is an appropriate constant, defined in Section 4. The iterative process is defined as follows:

$$\boldsymbol{y}^{[0]}(x) \equiv 0, \quad 0 < x < 1, \qquad \boldsymbol{y}^{[0]}(0) = \boldsymbol{y}(0), \quad \boldsymbol{y}^{[0]}(1) = \boldsymbol{y}(1).$$

For $k \geq 1$, the iterates $\boldsymbol{y}^{[k]}(x)$ are defined by

$$\boldsymbol{y}^{[k]}(x) = \begin{cases} \boldsymbol{y}_c^{[k]}(x) & \text{for} \quad x \in \bar{\Omega}_c, \\ \boldsymbol{y}_r^{[k]}(x) & \text{for} \quad x \in \bar{\Omega}_r \backslash \bar{\Omega}_c, \end{cases}$$

where $\boldsymbol{y}_p^{[k]}, \quad p = \{c, r\}$ are the solutions of the problems

$$\begin{array}{rcl} \mathbf{L}\boldsymbol{y}_r^{[k]}(x) & = & \mathbf{f} \text{ in } \Omega_r, \quad \boldsymbol{y}_r^{[k]}(1-2\tau) = \boldsymbol{y}^{[k-1]}(1-2\tau), \quad \boldsymbol{y}_r^{[k]}(1) = \boldsymbol{y}(1) \\ \mathbf{L}\boldsymbol{y}_c^{[k]}(x) & = & \mathbf{f} \text{ in } \Omega_c, \quad \boldsymbol{y}_c^{[k]}(0) = \boldsymbol{y}(0), \quad \boldsymbol{y}_c^{[k]}(1-\tau) = \boldsymbol{y}_r^{[k]}(1-\tau). \end{array}$$

Letting $\Omega_p = (d, e), \quad \bar{\Omega}_p = [d, e], \quad p = \{c, r\}$, note that the BVPs (1.6)–(1.7) satisfies the following maximum principle on each $\bar{\Omega}_p$.

Theorem 1. (Maximum principle). *Consider the BVPs (1.6)–(1.7). Let* $y_1(d) \geq 0, \quad y_2(d) \geq 0$, *and* $y_1(e) \geq 0$ *and* $y_2(e) \geq 0$, $L_1\boldsymbol{y}(x) \geq 0$, *and* $L_2\boldsymbol{y}(x) \geq 0$, *for* $x \in \Omega_p$. *Then,* $\boldsymbol{y}(x) \geq 0, \ \forall x \in \bar{\Omega}_p$.

Proof. Define the test functions $\boldsymbol{s}(x) = (s_1(x), s_2(x))^T$ by

$$s_1(x) = 5 - x^2, \quad s_2(x) = 1 + x, \quad x \in \bar{\Omega}_p.$$

Clearly, $s_1(d) > 0, \quad s_2(d) > 0, \ s_1(e) > 0, \quad s_2(e) > 0$. We can easily prove that $L_1\boldsymbol{s}(x) > 0$ and $L_2\boldsymbol{s}(x) > 0$, for $x \in \Omega_p$. Assume that the theorem is not true. We define

$$\xi = \max\Big\{ \max_{x \in \bar{\Omega}_p} \big(-y_1/s_1\big)(x), \ \max_{x \in \bar{\Omega}_p} \big(-y_2/s_2\big)(x) \Big\}.$$

Then, $\xi > 0$. Also, $(y_1 + \xi s_1)(x) \geq 0$, $(y_2 + \xi s_2)(x) \geq 0$, $\forall x \in \bar{\Omega}_p$. Furthermore, there exists a point $x_0 \in \Omega_p$ such that either

$$(y_1 + \xi s_1)(x_0) = 0 \quad \text{or} \quad (y_2 + \xi s_2)(x_0) = 0 \quad \text{or both.}$$

Case 1: $(y_1+\xi s_1)(x_0)=0$, for $x_0\in\Omega_p$. This implies that $y_1+\xi s_1$ attains its minimum at $x=x_0$. Therefore,

$$0<L_1(\boldsymbol{y}+\xi\boldsymbol{s})(x_0)=-(y_1+\xi s_1)^{''}(x_0)-(y_2+\xi s_2)(x_0)\le 0,$$

which is a contradiction.

Case 2: $(y_2+\xi s_2)(x_0)=0$, for $x_0\in\Omega_p$. This implies that $y_2+\xi s_2$ attains its minimum at $x=x_0$. Therefore,

$$\begin{aligned}0<L_2(\boldsymbol{y}+\xi\boldsymbol{s})(x_0)&=-\varepsilon(y_2+\xi s_2)^{''}(x_0)+a(x)(y_2+\xi s_2)'(x_0)\\&\quad+b(x)(y_2+\xi s_2)(x_0)+c(x)(y_1+\xi s_1)(x_0)\le 0,\end{aligned}$$

which is a contradiction. Hence it can be conclude that $\boldsymbol{y}(x)\ge 0$, $\forall x\in\bar{\Omega}$. □

An immediate consequence of this is the following stability result.

Lemma 1. (Stability result).*If $\boldsymbol{y}(x)$ is the solution of the BVPs (1.6)–(1.7) then $\forall x\in\bar{\Omega}_p$*

$$\|\boldsymbol{y}\|\le C\max\{|y_1(d)|,\,|y_2(d)|,\,|y_1(e)|,\,|y_2(e)|,\,\max_{x\in\Omega_p}|L_1\boldsymbol{y}(x)|,\,\max_{x\in\Omega_p}|L_2\boldsymbol{y}(x)|\}.$$

Proof. Set

$$M=C\max\{|y_1(d)|,\,|y_2(d)|,\,|y_1(e)|,\,|y_2(e)|,\,\max_{x\in\Omega_p}|L_1\boldsymbol{y}(x)|,\,\max_{x\in\Omega_p}|L_2\boldsymbol{y}(x)|\}.$$

Define two barrier functions $\boldsymbol{w}^{\pm}(x)=(w_1^{\pm}(x),\,w_2^{\pm}(x))^T$ by

$$w_1^{\pm}(x)=M(5-x^2)\pm y_1(x)\quad\text{and}\quad w_2^{\pm}(x)=M(1+x).$$

For $x\in\Omega_c$, we have

$$\begin{aligned}L_1\boldsymbol{w}^{\pm}(x)&=-w_1^{\pm\prime\prime}(x)-w_2^{\pm}(x)>M\tau\pm L_1\boldsymbol{y}(x)\ge 0,\\L_2\boldsymbol{w}^{\pm}(x)&=-\varepsilon w_2^{\pm\prime\prime}(x)+a(x)w_2^{\pm\prime}(x)+b(x)w_2^{\pm}(x)+c(x)w_1^{\pm}(x),\\&>M(\alpha-5\gamma)\pm L_2\boldsymbol{y}(x)\ge 0,\end{aligned}$$

by a proper choice of C. For $x\in\Omega_r$, we have

$$\begin{aligned}L_1\boldsymbol{w}^{\pm}(x)&=-w_1^{\pm\prime\prime}(x)-w_2^{\pm}(x)=M(1-x)\pm L_1\boldsymbol{y}(x)\ge 0,\\L_2\boldsymbol{w}^{\pm}(x)&=-\varepsilon w_2^{\pm\prime\prime}(x)+a(x)w_2^{\pm\prime}(x)+b(x)w_2^{\pm}(x)+c(x)w_1^{\pm}(x),\\&>M(\alpha-5\gamma)\pm L_2\boldsymbol{y}(x)\ge 0,\end{aligned}$$

by a proper choice of C. Furthermore, we have

$$\begin{aligned}&w_1^{\pm}(d)=w_1^{\pm}(0)=5M\pm y_1(0)\ge 0,\quad w_2^{\pm}(d)=w_2^{\pm}(0)=M\pm y_2(0)\ge 0,\\&w_1^{\pm}(e)=w_1^{\pm}(1-\tau)>3M\pm y_1(1-\tau)\ge 0,\\&w_2^{\pm}(e)=w_2^{\pm}(1-\tau)>M\pm y_2(1-\tau)\ge 0,\\&w_1^{\pm}(d)=w_1^{\pm}(1-2\tau)>4M\pm y_1(1-2\tau)\ge 0,\\&w_2^{\pm}(d)=w_2^{\pm}(1-2\tau)>M\pm y_2(1-2\tau)\ge 0,\\&w_1^{\pm}(e)=w_1^{\pm}(1)=4M\pm y_1(1)\ge 0,\quad w_2^{\pm}(e)=w_2^{\pm}(1)=2M\pm y_2(1)\ge 0\end{aligned}$$

by a proper choice of C. Applying Theorem 1 to the barrier functions $\boldsymbol{w}^{\pm}(x)$, we get the desired result. □

3 Estimates of derivatives

In Section 5 we establish the convergence of the discrete Schwarz method described in Section 4. To prove convergence of the numerical solution, we need the following stronger results on the estimates of the derivatives of the components of the solution of the BVPs (1.6)–(1.7). Now, decompose the solution $\boldsymbol{y}(x)$ of (1.6)–(1.7) into smooth and singular components $\boldsymbol{v}(x)$ and $\boldsymbol{w}(x)$ respectively as

$$\boldsymbol{y}(x) = \boldsymbol{v}(x) + \boldsymbol{w}(x), \tag{3.1}$$

where $\boldsymbol{v}(x) = (v_1(x), v_2(x))^T$ is the solution of the reduced problem of the BVPs (1.6)–(1.7) given by

$$\begin{cases} -v_1''(x) - v_2(x) = 0, \\ a(x)v_2'(x) + b(x)v_2(x) + c(x)v_1(x) = f(x), \\ v_1(0) = q_1, \quad v_1(1) = q_3, \quad v_2(0) = q_2 \end{cases} \tag{3.2}$$

and $\boldsymbol{w}(x) = (w_1(x), w_2(x))^T$ is a layer correction term given by

$$\begin{cases} w_1(x) = -(\varepsilon/a(0))^2(q_4 - v_2(1))e^{-a(0)(1-x)/\varepsilon}, \\ w_2(x) = (q_4 - v_2(1))e^{-a(0)(1-x)/\varepsilon} \end{cases}$$

and $\boldsymbol{w(x)}$ satisfies

$$\begin{cases} -w_1''(x) - w_2(x) = 0, \\ -\varepsilon w_2''(x) + a(0)w_2'(x) = 0, \\ w_1(0) = w_1(1)e^{-a(0)/\varepsilon}, \quad w_1(1) = -w_2(1)(\varepsilon/a(0))^2, \\ w_2(0) = w_2(1)e^{-a(0)/\varepsilon}, \quad w_2(1) = q_4 - v_2(1). \end{cases} \tag{3.3}$$

The following lemma gives estimates of the derivatives of these components.

Lemma 2. *The solution $\boldsymbol{y}(x)$ of the BVPs (1.6)–(1.7) has the decomposition $\boldsymbol{y}(x) = \boldsymbol{v}(x) + \boldsymbol{w}(x)$ into smooth and singular components, satisfy*

$$\begin{aligned} &|\boldsymbol{v}_1^{(l)}(x)| \le C, \quad |\boldsymbol{v}_2^{(l)}(x)| \le C, \\ &|\boldsymbol{w}_1^{(l)}(x)| \le C\varepsilon^{-(l-2)}e^{-\alpha(1-x)/\varepsilon}, \quad |\boldsymbol{w}_2^{(l)}(x)| \le C\varepsilon^{-(l)}e^{-\alpha(1-x)/\varepsilon}, \end{aligned}$$

for $0 \le l \le 4$, $\forall x \in \bar{\Omega} = (\bar{\Omega}_r \backslash \bar{\Omega}_c) \cup \bar{\Omega}_c$, $\boldsymbol{v}(x)$ and $\boldsymbol{w}(x)$ are given by (3.2)–(3.3).

Proof. It is easy to check that

$$|v_1^{(l)}(x)| \le C \quad \text{and} \quad |v_2^{(l)}(x)| \le C \quad \text{for} \quad x \in \bar{\Omega}$$

as $a(x)$, $b(x)$, $c(x)$, $f(x)$ are sufficiently smooth functions. Differentiating the equation (3.3) l times and using the method of induction one can get

$$\begin{aligned} |w_1^{(l)}(x)| &\le C\varepsilon^{-(l-2)}\exp(-\alpha(1-x)/\varepsilon), \\ |w_2^{(l)}(x)| &\le C\varepsilon^{-(l)}\exp(-\alpha(1-x)/\varepsilon). \end{aligned}$$

□

4 Discrete Schwarz method

The continuous overlapping Schwarz method described in Section 2 is discretized by introducing uniform meshes on each subdomain. The domain $\Omega = (0,1)$ is divided into two overlapping subdomains as $\Omega_c = (0, 1-\tau)$ and $\Omega_r = (1-2\tau, 1)$. The subdomain transition parameter τ is chosen to be the Shishkin transition point $\tau = \min\left\{\frac{1}{3}, \frac{4\varepsilon}{\alpha}\ln N\right\}$ as in ([10], p.91). In each subdomain, $\Omega_p = (d,e), \quad p = \{c,r\}$, construct a uniform mesh $\bar{\Omega}_p^N = \{d = x_0 < x_1 < x_2 < \cdots < x_n = e\}$ with $h_p = (x_i - x_{i-1})/N = (e-d)/N$.

In the proposed scheme we use the central finite difference scheme with a uniform mesh on the subdomain Ω_r and the mid-point difference scheme with a uniform mesh on the subdomain Ω_c. Then in each subdomain $\Omega_p^N, \ p = \{c,r\}$, the corresponding discretization is,

$$\mathbf{L}^N\mathbf{Y}_c(x_i) = \begin{cases} L_1^N\mathbf{Y}_c(x_i) = -\delta^2 Y_{1,c}(x_i) - \hat{Y}_{2,c}(x_i) = 0, \quad i = 1,\dots,N-1, \\ L_2^N\mathbf{Y}_c(x_i) = -\varepsilon\delta^2 Y_{2,c}(x_i) + a_{i-1/2}D^- Y_{2,c}(x_i) + c_{i-1/2}\hat{Y}_{1,c}(x_i) \\ \qquad\qquad + b_{i-1/2}\hat{Y}_{2,c}(x_i) = f_{i-1/2}, \quad i = 1,\dots,N-1, \end{cases}$$

$$\mathbf{L}^N\mathbf{Y}_r(x_i) = \begin{cases} L_1^N\mathbf{Y}_r(x_i) = -\delta^2 Y_{1,r}(x_i) - Y_{2,r}(x_i) = 0, \quad i = 1,\dots,N-1, \\ L_2^N\mathbf{Y}_r(x_i) = -\varepsilon\delta^2 Y_{2,r}(x_i) + a_i D^0 Y_{2,r}(x_i) + b_i Y_{2,r}(x_i) \\ \qquad\qquad + c_i Y_{1,r}(x_i) = f_i, \quad i = 1,\dots,N-1, \end{cases}$$

where

$$\delta^2 Y_{j,p}(x_i) = \frac{1}{h_p^2}\left(Y_{j,p}(x_{i+1}) - 2Y_{j,p}(x_i) + Y_{j,p}(x_{i-1})\right),$$

$$D^- Y_{j,c}(x_i) = \frac{Y_{j,c}(x_i) - Y_{j,c}(x_{i-1})}{h_c}, \quad \hat{Y}_{j,c}(x_i) \equiv (Y_{j,c}(x_i) + Y_{j,c}(x_{i-1}))/2,$$

$$D^0 Y_{j,r}(x_i) = \frac{Y_{j,r}(x_{i+1}) - Y_{j,r}(x_{i-1})}{2h_r}, \ a_{i-1/2} \equiv a((x_{i-1} + x_i)/2), \ a_i \equiv a(x_i),$$

similarly for $b_{i-1/2}$, $c_{i-1/2}$, $f_{i-1/2}$, b_i, c_i and f_i, $j = 1,2$.

The discrete problem is $\mathbf{L}^N\mathbf{Y}_p(x_i) = \mathbf{f}(x_i)$, where

$$\mathbf{f}(x_i) = \begin{cases} \mathbf{f}_{i-\frac{1}{2}}, \quad x_i \in \bar{\Omega}_c^N, \\ \mathbf{f}_i, \quad x_i \in \bar{\Omega}_r^N. \end{cases}$$

Then the algorithm for discrete Schwarz method is defined as follows.

Step1: We choose the initial mesh function

$$\mathbf{Y}^{[0]}(x_i) \equiv 0, \quad 0 < x_i < 1, \quad \mathbf{Y}^{[0]}(0) = \boldsymbol{y}(0), \quad \mathbf{Y}^{[0]}(1) = \boldsymbol{y}(1).$$

Step2: We compute the mesh functions ${\mathbf{Y}_p}^{[k]}$, $p = \{r,c\}$ which are the solutions of the following discrete problems

$$\mathbf{L}^N\mathbf{Y}_r^{[k]}(x_i) = \mathbf{f}_i, \ \ x_i \in \Omega_r^N, \ \ \mathbf{Y}_r^{[k]}(1-2\tau) = \bar{\mathbf{Y}}^{[k-1]}(1-2\tau), \ \ \mathbf{Y}_r^{[k]}(1) = \boldsymbol{y}(1),$$

$$\mathbf{L}^N\mathbf{Y}_c^{[k]}(x_i) = \mathbf{f}_{i-\frac{1}{2}}, \ \ x_i \in \Omega_c^N, \quad \mathbf{Y}_c^{[k]}(0) = \boldsymbol{y}(0), \quad \mathbf{Y}_c^{[k]}(1-\tau) = \bar{\mathbf{Y}}_r^{[k]}(1-\tau),$$

where $\bar{\boldsymbol{Y}}^{[k]}$, $k \geq 1$ denotes the piecewise linear interpolant of $\boldsymbol{Y}^{[k]}$ on the mesh $\bar{\Omega}^N := (\bar{\Omega}_r^N \backslash \bar{\Omega}_c) \cup \bar{\Omega}_c^N$.

Step3: We compute the mesh function $\mathbf{Y}^{[k]}$ by combining together the solutions on the subdomains

$$\mathbf{Y}^{[k]}(x_i) = \begin{cases} \mathbf{Y}_c^{[k]}(x_i), & \text{for } \ x_i \in \bar{\Omega}_c^N, \\ \mathbf{Y}_r^{[k]}(x_i), & \text{for } \ x_i \in \bar{\Omega}_r^N \backslash \bar{\Omega}_c. \end{cases}$$

Step4: If the stopping criterion $\|\mathbf{Y}^{[k+1]} - \mathbf{Y}^{[k]}\|_{\bar{\Omega}^N} \leq tol$ is reached, then stop; otherwise go to Step 2. Here *tol* is the user prescribed accuracy. For each $p = \{c, r\}$, the matrix associated with $\mathbf{L}^N$ is M-matrix, and hence it satisfies the following discrete maximum principle.

Lemma 3. ***(Discrete maximum principle)*** *Assume that* $\boldsymbol{Y}(x_0) \geq \boldsymbol{0}$ *and* $\boldsymbol{Y}(x_N) \geq \boldsymbol{0}$, *then* $\boldsymbol{L}^N \boldsymbol{Y}(x_i) \geq \boldsymbol{0}$, $\forall x_i \in \Omega_p^N$ *implies that* $\boldsymbol{Y}(x_i) \geq \boldsymbol{0}$, $\forall x_i \in \bar{\Omega}_p^N$.

Proof. Please refer to [11, 12] and [19]. □

An immediate consequence of this lemma is the following stability result.

Lemma 4. *If* $Y_j(x_i)$ *is any mesh function then for all* $x_i \in \bar{\Omega}_p^N$

$$|Y_j(x_i)| \leq C \max\{|Y_1(x_0)|, |Y_1(x_N)|, |Y_2(x_0)|, |Y_2(x_N)|, \|L_1^N \boldsymbol{Y}\|_{\Omega_p^N}, \|L_2^N \boldsymbol{Y}\|_{\Omega_p^N}\}$$

for $j=1, 2$

Proof. Please refer to [11, 12] and [19]. □

5 Error estimates

In this Section, we estimate the error in discrete Schwarz iterates and prove that two iterations are required to attain almost second-order convergence. Following the method of analysis adapted in [18] and [20] we derive error estimates. The analysis proceeds as follows.

Lemma 5. *Let* $\boldsymbol{y}$ *be the solution of (1.6)–(1.7) and let* $\mathbf{Y}^{[k]}$ *be the* k^{th} *iterate of the discrete Schwarz method described as in Section 4. Then, there are constants* C *such that*

$$\|\boldsymbol{Y}^{[k]} - \boldsymbol{y}\|_{\bar{\Omega}^N} \leq C2^{-k} + CN^{-2} \ln^3 N.$$

Proof. At the first iteration $(\mathbf{Y}^{[0]} - \boldsymbol{y})(0) = \mathbf{0}$ and $(\mathbf{Y}^{[0]} - \boldsymbol{y})(1) = \mathbf{0}$. Since $\mathbf{Y}^{[0]}(x_i) = \mathbf{0}$ for $x_i \in \Omega^N := \{x_1 < x_2 < x_3 \cdots < x_{N-1}\}$ we can use Lemma 1 to show that

$$\|\mathbf{Y}^{[0]} - \boldsymbol{y}\|_{\Omega^N} = \|\boldsymbol{y}\|_{\Omega^N} \leq C.$$

Clearly, there are constants C such that

$$\|\mathbf{Y}^{[0]} - \boldsymbol{y}\|_{\bar{\Omega}^N} \leq C2^0 + CN^{-2} \ln^3 N.$$

Thus, the result holds for $k = 0$ and the proof is now completed by induction. Assume that, for an arbitrary integer $k \geq 0$, there exists C such that

$$\|\mathbf{Y}^{[k]} - \boldsymbol{y}\|_{\bar{\Omega}^N} \leq C2^{-k} + CN^{-2}\ln^3 N.$$

Case (i): Error bound estimation on $\bar{\Omega}_r^N$. In the proposed scheme we use the central finite difference scheme on $\bar{\Omega}_r^N$. One can deduce the following truncation error estimate as in [12] on $x_i \in \bar{\Omega}_r^N$ as

$$\|\mathbf{L}^N(\mathbf{Y} - \boldsymbol{y})\|_{\Omega_r^N} \leq \begin{pmatrix} Ch_r^2\|y_1^{(4)}\|_{\Omega_r^N} \\ C\varepsilon h_r^2\|y_2^{(4)}\|_{\Omega_r^N} + Ch_r^2\|y_2^{(3)}\|_{\Omega_r^N} \end{pmatrix}. \tag{5.1}$$

In order to find a bound on $\|\mathbf{L}^N(\mathbf{Y}_r^{[k+1]} - \boldsymbol{y})\|_{\Omega_r^N}$ we must decompose $\boldsymbol{y}$ as in (3.1). Consider

$$\begin{aligned} \|\mathbf{L}^N(\mathbf{Y}_r^{[k+1]} - \boldsymbol{y})\|_{\Omega_r^N} &= \|\mathbf{f} - \mathbf{L}\boldsymbol{y}\|_{\Omega_r^N} = \|(\mathbf{L}^N - \mathbf{L})\boldsymbol{y}\|_{\Omega_r^N} \\ &\leq \|(\mathbf{L}^N - \mathbf{L})\boldsymbol{v}\|_{\Omega_r^N} + \|(\mathbf{L}^N - \mathbf{L})\boldsymbol{w}\|_{\Omega_r^N}. \end{aligned} \tag{5.2}$$

For the first term on the right-hand side of (5.2), we use the local truncation error estimate (5.1), $h_r \leq CN^{-1}$, $\varepsilon \leq CN^{-1}$, and Lemma 2 to get

$$\begin{aligned} \|(\mathbf{L}^N - \mathbf{L})\boldsymbol{v}\|_{\Omega_r^N} &\leq \begin{pmatrix} Ch_r^2\|v_1^{(4)}\|_{\Omega_r^N} \\ C\varepsilon h_r^2\|v_2^{(4)}\|_{\Omega_r^N} + Ch_r^2\|v_2^{(3)}\|_{\Omega_r^N} \end{pmatrix} \\ &\leq \begin{pmatrix} CN^{-2} \\ CN^{-3} + CN^{-2} \end{pmatrix} \leq CN^{-2}. \end{aligned}$$

For the second term on the right-hand side of (5.2), when $\tau = \frac{4\varepsilon}{\alpha}\ln N$, using the local truncation error estimate (5.1), and $h_r \leq C\varepsilon N^{-1}\ln N$, we have

$$\begin{aligned} \|(\mathbf{L}^N - \mathbf{L})\boldsymbol{w}\|_{\Omega_r^N} &\leq \begin{pmatrix} Ch_r^2\|w_1^{(4)}\|_{\Omega_r^N} \\ C\varepsilon h_r^2\|w_2^{(4)}\|_{\Omega_r^N} + Ch_r^2\|w_2^{(3)}\|_{\Omega_r^N} \end{pmatrix} \\ &\leq \begin{pmatrix} Ch_r^2\varepsilon^{-2} \\ Ch_r^2\varepsilon^{-3} \end{pmatrix} \leq C\varepsilon^{-1}N^{-2}\ln^2 N. \end{aligned}$$

Using the above estimates in (5.2), we have

$$\|\mathbf{L}^N(\mathbf{Y}_r^{[k+1]} - \boldsymbol{y})\|_{\Omega_r^N} \leq CN^{-2}\ln^3 N + C\varepsilon^{-1}N^{-2}\ln^2 N$$

for some C. The end point of the subdomain Ω_r^N is $1 - 2\tau$, which is in general is not in $\Omega^N = \{x_1 < x_2 < x_3 < \ldots < x_{N-1}\}$, so we use a piecewise linear interpolant of the previous iterate to determine $\mathbf{Y}_r^{[k+1]}(1 - 2\tau)$. Now, using our inductive argument, we have

$$\begin{aligned} |(\mathbf{Y}_r^{[k+1]} - \boldsymbol{y})(1 - 2\tau)| &= |(\bar{\mathbf{Y}}^{[k]} - \boldsymbol{y})(1 - 2\tau)| = |(\mathbf{Y}^{[k]} - \boldsymbol{y})(1 - 2\tau)| \\ &\leq |(\mathbf{Y}^{[k]} - \bar{\boldsymbol{y}})(1 - 2\tau)| + |(\bar{\boldsymbol{y}} - \boldsymbol{y})(1 - 2\tau)|, \end{aligned} \tag{5.3}$$

where $\bar{\boldsymbol{y}}$ is the piecewise linear interpolant of $\boldsymbol{y}$ using grid points of $\bar{\Omega}_c^N$. For the second term on the right-hand side of (5.3), using solution decomposition $\boldsymbol{y}$ as in (3.1), we get

$$|(\bar{\boldsymbol{y}}-\boldsymbol{y})(1-2\tau)| \leq |(\bar{\boldsymbol{v}}-\boldsymbol{v})(1-2\tau)| + |(\bar{\boldsymbol{w}}-\boldsymbol{w})(1-2\tau)|. \tag{5.4}$$

Note that $(1-2\tau)$ lies in $\bar{\Omega}_c$. For any $\mathbf{z} \in C^2(\bar{\Omega}_c)$, standard argument of piecewise linear interpolant $\bar{\mathbf{z}}$ gives

$$|(\mathbf{z}-\bar{\mathbf{z}})(1-2\tau)| \leq Ch_c^2\|\mathbf{z}^{(2)}\|_{\bar{\Omega}_c} \quad \text{and} \quad |(\mathbf{z}-\bar{\mathbf{z}})(1-2\tau)| \leq C\|\mathbf{z}\|_{\bar{\Omega}_c}. \tag{5.5}$$

For the first term on the right-hand side of (5.4), we use the first bound of (5.5), $h_c \leq CN^{-1}$, and Lemma 2 to get

$$|(\bar{\boldsymbol{v}}-\boldsymbol{v})(1-2\tau)| \leq Ch_c^2\|\boldsymbol{v}^{(2)}\|_{\bar{\Omega}_c} \leq CN^{-2}.$$

For the second term on the right-hand side of (5.4), when $\tau = \frac{4\varepsilon}{\alpha}\ln N$, note that the layer function $\boldsymbol{w}$ is monotonically increasing in the region $(1/3, 1-\tau) \subset \bar{\Omega}_c$. Hence using the second bound of (5.5), we have

$$|(\bar{\boldsymbol{w}}-\boldsymbol{w})(1-2\tau)| \leq C\|\boldsymbol{w}\|_{\bar{\Omega}_c}. \tag{5.6}$$

Now, using (5.6) in (5.3) we have

$$\begin{aligned} |(\mathbf{Y}_r^{[k+1]}-\boldsymbol{y})(1-2\tau)| &\leq C2^{-k} + CN^{-2}\ln^3 N + CN^{-2} \\ &\leq C2^{-k} + CN^{-2}\ln^3 N. \end{aligned}$$

Consider the mesh function

$$\begin{aligned} \Psi^{\pm}(x_i) =& \mathbf{C}\left(\frac{3+x_i}{2}\right)2^{-k} + \mathbf{C}(1+x_i)N^{-2}\ln^3 N \\ &+ \mathbf{C}(x_i-(1-2\tau))\varepsilon^{-1}N^{-2}\ln^2 N \pm (\mathbf{Y}_r^{[k+1]}-\boldsymbol{y})(x_i), \end{aligned}$$

where $\mathbf{C}$ is positive constants suitably chosen so that the following are satisfied.

Note that, $\Psi^{\pm}(1-2\tau) > 0$, $\Psi^{\pm}(1) > 0$ and $L^N\Psi^{\pm}(x_i) > 0$. Using the discrete maximum principle for the operator L^N on $\bar{\Omega}_r^N$ we get,

$$\begin{aligned} \|(\mathbf{Y}_r^{[k+1]}-\boldsymbol{y})\|_{\bar{\Omega}_r^N} \leq& C\left(\frac{3+x_i}{2}\right)2^{-k} + C(1+x_i)N^{-2}\ln^3 N \\ &+ C(x_i-(1-2\tau))\varepsilon^{-1}N^{-2}\ln^2 N. \end{aligned}$$

Consequently,

$$\begin{aligned} \|(\mathbf{Y}_r^{[k+1]}-\boldsymbol{y})\|_{\bar{\Omega}_r^N\setminus\bar{\Omega}_c} &\leq 4C\left(\frac{1}{2}\right)2^{-k} + 2CN^{-2}\ln^3 N + 2C\tau\varepsilon^{-1}N^{-2}\ln^2 N \\ &\leq C2^{-(k+1)} + +CN^{-2}\ln^3 N + C\tau\varepsilon^{-1}N^{-2}\ln^2 N. \end{aligned}$$

But since $\tau = \frac{4\varepsilon}{\alpha}\ln N$, this gives

$$\|(\mathbf{Y}_r^{[k+1]}-\boldsymbol{y})\|_{\bar{\Omega}_r^N\setminus\bar{\Omega}_c} \leq C2^{-(k+1)} + CN^{-2}\ln^3 N. \tag{5.7}$$

Case (ii): Error bound estimation on $\bar{\Omega}_c^N$. We use solution decomposition as in Lemma 2 at each point $x_i \in \bar{\Omega}_c^N$, the difference $(\mathbf{Y}_c^{[k+1]} - \boldsymbol{y})$ can be written in the form

$$(\mathbf{Y}_c^{[k+1]} - \boldsymbol{y})(x_i) = (\mathbf{V}_c^{[k+1]} - \boldsymbol{v})(x_i) + (\mathbf{W}_c^{[k+1]} - \boldsymbol{w})(x_i). \tag{5.8}$$

Suppose that $(1-\tau)$ lies in $\bar{\Omega}_r$. For any $\mathbf{z} \in \mathbf{C}^2(\bar{\Omega}_r)$, standard argument of piecewise linear interpolant $\bar{\mathbf{z}}$ gives

$$|(\mathbf{z} - \bar{\mathbf{z}})(1-\tau)| \leq Ch_r^2 \|\mathbf{z}^{(2)}\|_{\bar{\Omega}_r}. \tag{5.9}$$

In the proposed scheme we use the mid-point difference scheme on $\bar{\Omega}_c^N$. One can deduce the following truncation error estimate as in [12] on $x_i \in \bar{\Omega}_c^N$ as

$$\|(\mathbf{L}^N - \mathbf{L})\boldsymbol{y}\|_{\Omega_c^N} \leq \begin{pmatrix} Ch_c^2\|y_1^{(4)}\|_{\Omega_c^N} + Ch_c^2\|y_2^{(2)}\|_{\Omega_c^N} \\ C\varepsilon h_c^2\|y_2^{(4)}\|_{\Omega_c^N} + Ch_c^2(\|y_2^{(3)}\|_{\Omega_c^N} + \|y_1^{(2)}\|_{\Omega_c^N}) \end{pmatrix}.$$

Subcase (i): For the first term on the right-hand side of (5.8), using the above local truncation error estimate, $h_c \leq CN^{-1}$, $\varepsilon \leq CN^{-1}$ and Lemma 2, we get

$$\begin{aligned} &\|\mathbf{L}^N(\mathbf{V}_c^{(k+1)} - \boldsymbol{v})\|_{\Omega_c^N} = \|\mathbf{f} - \mathbf{L}\boldsymbol{v}\|_{\Omega_c^N} = \|(\mathbf{L}^N - \mathbf{L})\boldsymbol{v}\|_{\Omega_c^N} \\ &\leq \begin{pmatrix} Ch_c^2\|v_1^{(4)}\|_{\Omega_c^N} + Ch_c^2\|v_2^{(2)}\|_{\Omega_c^N} \\ C\varepsilon h_c^2\|v_2^{(4)}\|_{\Omega_c^N} + Ch_c^2(\|v_2^{(3)}\|_{\Omega_c^N} + \|v_1^{(2)}\|_{\Omega_c^N}) \end{pmatrix} \leq \begin{pmatrix} CN^{-2} \\ CN^{-2} \end{pmatrix} \leq CN^{-2}. \end{aligned}$$

Now, using our inductive argument, the bound of (5.9), $h_r \leq CN^{-1}$, $\varepsilon \leq CN^{-1}$, and Lemma 2, we get

$$\begin{aligned} |(\mathbf{V}_c^{[k+1]} - \boldsymbol{v})(1-\tau)| &= |(\bar{\mathbf{V}}_r^{[k+1]} - \boldsymbol{v})(1-\tau)| = |(\bar{\mathbf{V}} - \boldsymbol{v})(1-\tau)| \\ &\leq Ch_r^2\|\boldsymbol{v}^{(2)}\|_{\bar{\Omega}_r} \leq CN^{-2}, \end{aligned}$$

where we have used the fact that $(1-\tau)$ is the mesh point of $\bar{\Omega}_r^N$.

Consider the mesh function

$$\Phi^{\pm}(x_i) = \mathbf{C}\left(\frac{x_i}{2(1-\tau)}\right)2^{-k} + (1+x_i)\mathbf{C}N^{-2} \pm (\mathbf{V}_c^{[k+1]} - \boldsymbol{v})(x_i),$$

where C is positive constants to be choosen suitably, so that the following expressions are satisfied. Note that $\Phi^{\pm}(0) > 0$, $\Phi^{\pm}(1-\tau) > 0$, $L^N\Phi^{\pm}(x_i) > 0$.

We use the discrete maximum principle for the operator L^N on $\bar{\Omega}_c^N$ to get

$$\begin{aligned} \|\mathbf{V}_c^{[k+1]} - \boldsymbol{v}\|_{\bar{\Omega}_c^N} &\leq C\left(\frac{1}{2}\right)2^{-k} + C(2-\tau)N^{-2} \\ &\leq C2^{-(k+1)} + CN^{-2}. \end{aligned}$$

Subcase (ii): For the second term on the right-hand side of (5.8), when $\tau = \frac{4\varepsilon}{\alpha}\ln N$, using the arguments discussed as in ([11], Lemma 6) for $x_i \in \Omega_c^N$ we get

$$\|\mathbf{W}_c^{[k+1]} - \boldsymbol{w}\|_{\Omega_c^N} \leq CN^{-2}.$$

Now, using error bound for the smooth and layer parts we get

$$\|(\mathbf{Y}_c^{[k+1]} - \boldsymbol{y})\|_{\Omega_c^N} \le C2^{-(k+1)} + CN^{-2}\ln^3 N. \tag{5.10}$$

Combining the error bounds (5.7) and (5.10), we have

$$\|\mathbf{Y}^{[k+1]} - \boldsymbol{y}\|_{\bar{\Omega}^N} \le C2^{-(k+1)} + CN^{-2}\ln^3 N.$$

This completes the proof. □

Now we will show that the discrete Schwarz iterates converge at a higher rate than that suggested by Lemma 5.

Lemma 6. *Let $\mathbf{Y}^{[k]}(x_i)$ be the k^{th} iterate of the discrete Schwarz method described in Section 4. Then there exists some C such that*

$$\| \mathbf{Y}^{[k+1]} - \mathbf{Y}^{[k]}\|_{\bar{\Omega}^N} \le C\nu^k, \quad \textit{where} \;\; \nu = \left(1 + \frac{\tau\alpha}{2\varepsilon N}\right)^{-N/2} < 1.$$

Furthermore, if $\tau = \frac{4\varepsilon}{\alpha}\ln N$, then $\nu \le 2N^{-1}$.

Proof. At the first iteration $\|\mathbf{Y}^{[0]}\|_{\Omega^N} = \mathbf{0}$. Then clearly

$$\|\mathbf{Y}^{[1]} - \mathbf{Y}^{[0]}\|_{\Omega^N} = \|\mathbf{Y}^{[1]}\|_{\Omega^N}.$$

$\mathbf{Y}_r^{[1]}$ satisfies

$$\mathbf{L}^N\mathbf{Y}_r^{[1]} = \mathbf{f}_i \quad \text{for} \quad x_i \in \Omega_r^N,$$
$$\mathbf{Y}_r^{[1]}(1-2\tau) = \bar{\mathbf{Y}}^{[0]}(1-2\tau), \quad \mathbf{Y}_r^{[1]}(1) = \boldsymbol{y}(1).$$

Therefore, we use Lemma 4 to obtain $\|\mathbf{Y}_r^{[1]}\|_{\bar{\Omega}_r^N} \le C$.

Consequently, $\|\mathbf{Y}_r^{[1]}\|_{\bar{\Omega}_r^N\setminus\bar{\Omega}_c} \le C$. Also $\mathbf{Y}_c^{[1]}$ satisfies

$$\mathbf{L}^N\mathbf{Y}_c^{[1]} = \mathbf{f}_{i-1/2} \;\text{ for }\; x_i \in \Omega_c^N,$$
$$\mathbf{Y}_c^{[1]}(0) = \boldsymbol{y}(0), \quad \mathbf{Y}_c^{[1]}(1-\tau) = \bar{\mathbf{Y}}_r^{[1]}(1-\tau).$$

Therefore, we can apply Lemma 4 to get $\|\mathbf{Y}_c^{[1]}\|_{\bar{\Omega}_c^N} \le C$. Combining all these estimates we obtain

$$\|\mathbf{Y}^{[1]} - \mathbf{Y}^{[0]}\|_{\bar{\Omega}^N} \le C\nu^0.$$

Thus, the result holds for $k = 0$ and the proof is now completed by induction argument. Assume that for an arbitrary integer $k \ge 0$

$$\|\mathbf{Y}^{[k+1]} - \mathbf{Y}^{[k]}\|_{\bar{\Omega}^N} \le \mathbf{C}\nu^k, \;\; \text{where} \;\; \nu = \left(1 + \frac{\alpha\tau}{2\varepsilon N}\right)^{-N/2}.$$

Note that $\mathbf{L}^N(\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]})(x_i) = \mathbf{0}$ for $x_i \in \Omega_c^N$, $(\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]})(0) = \mathbf{0}$, and $|(\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]})(1-\tau)| \le C\nu^k$.

Let $\Phi_c^{[k+1]}(x_i) = \left(\Phi_{1,c}^{[k+1]}(x_i), \Phi_{2,c}^{[k+1]}(x_i)\right)^T$ be the solution of

$$\begin{cases} A\delta^2\Phi_c^{[k+1]}(x_i) + \alpha D^-\Phi_c^{[k+1]}(x_i) + \beta\hat{\Phi}_c^{[k+1]}(x_i) = \mathbf{0} \quad \text{for} \quad x_i \in \Omega_c^N, \\ \Phi_c^{[k+1]}(x_0) = \mathbf{0}, \quad \Phi_c^{[k+1]}(x_n) = \mathbf{C}\nu^k, \end{cases} \tag{5.11}$$

where $A = \begin{pmatrix} -1 & 0 \\ 0 & -\varepsilon \end{pmatrix}$. Using the maximum principle argument we note that $\Phi_c^{[k+1]}(0) \geq \mathbf{0}$, $\Phi_c^{[k+1]}(1-\tau) \geq \mathbf{0}$, $\Phi_c^{[k+1]}(x_i) \geq \mathbf{0}$ for $x_i \in \bar{\Omega}_c^N$, and thus one can easily deduce that $L^N\Phi_c^{[k+1]}(x_i) \geq \mathbf{0}$, for $x_i \in \Omega_c^N$. Hence

$$\begin{aligned} &L^N(\Phi_c^{[k+1]} - (\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]}))(x_i) = L^N(\Phi_c^{[k+1]})(x_i) - L^N(\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]})(x_i), \\ &\qquad \geq \mathbf{0}, \text{ as } L^N(\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]})(x_i) = \mathbf{0} \ \text{ for } x_i \in \Omega_c^N, \\ &\Phi_c^{[k+1]}(0) - (\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]})(0) \geq \mathbf{0}, \ \Phi_c^{[k+1]}(1-\tau) - (\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]})(1-\tau) \geq \mathbf{0}. \end{aligned}$$

Then by using Lemma 3 we have

$$(\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]})(x_i) \leq \Phi_c^{[k+1]}(x_i) \ \text{ for } \quad x_i \in \Omega_c^N. \tag{5.12}$$

The exact solution to the difference problem (5.11) is

$$\Phi_c^{[k+1]}(x_i) = \mathbf{C}\nu^k(m_1^i - m_2^i)/(m_1^N - m_2^N),$$

where

$$\begin{aligned} m_1 &= \left(1 + \frac{\alpha h_c}{2\varepsilon} + \frac{\beta h_c^2}{4\varepsilon}\right) + \sqrt{\left(1 + \frac{\alpha h_c}{2\varepsilon} + \frac{\beta h_c^2}{4\varepsilon}\right)^2 + \left(-1 - \frac{\alpha h_c}{\varepsilon} + \frac{\beta h_c^2}{2\varepsilon}\right)} \\ &\geq 1 + \frac{\alpha h_c}{2\varepsilon} = \left(1 + \frac{\alpha(1-\tau)}{2\varepsilon N}\right) \geq \left(1 + \frac{\alpha\tau}{2\varepsilon N}\right), \\ m_2 &= \left(1 + \frac{\alpha h_c}{2\varepsilon} + \frac{\beta h_c^2}{4\varepsilon}\right) - \sqrt{\left(1 + \frac{\alpha h_c}{2\varepsilon} + \frac{\beta h_c^2}{4\varepsilon}\right)^2 + \left(-1 - \frac{\alpha h_c}{\varepsilon} + \frac{\beta h_c^2}{2\varepsilon}\right)}. \end{aligned}$$

Now

$$\mathbf{L}^N(\mathbf{Y}_r^{[k+2]} - \mathbf{Y}_r^{[k+1]})(x_i) = \mathbf{0}, \quad \forall \ x_i \in \bar{\Omega}_r^N, \quad (\mathbf{Y}_r^{[k+2]} - \mathbf{Y}_r^{[k+1]})(1) = \mathbf{0}.$$

Using our inductive hypothesis and (5.12)

$$\begin{aligned} |(\ \mathbf{Y}_r^{[k+2]} - \mathbf{Y}_r^{[k+1]})(1-2\tau)| &= |(\bar{\mathbf{Y}}_c^{[k+1]} - \bar{\mathbf{Y}}_c^{[k]})(1-2\tau)| \\ &= |(\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]})(1-2\tau)| \leq \Phi_c^{[k+1]}(1-2\tau), \end{aligned}$$

where we have used the fact that $(1-2\tau)$ is the mesh point of $\bar{\Omega}_c^N$. Using Lemma 4 we obtain

$$\|\mathbf{Y}_r^{[k+2]} - \mathbf{Y}_r^{[k+1]}\|_{\bar{\Omega}_r^N} \quad \leq \quad \Phi_c^{[k+1]}(1-2\tau).$$

Here we used

$$\Phi_c^{[k+1]}(1-2\tau) = \mathbf{C}\nu^k \frac{m_1^{N/2} - m_2^{N/2}}{m_1^N - m_2^N} \le \frac{\mathbf{C}\nu^k}{m_1^{N/2}}$$
$$= \mathbf{C}\nu^k \left(1 + \frac{\tau\alpha}{2\varepsilon N}\right)^{-N/2} = \mathbf{C}\nu^k \left(1 + \frac{\tau\alpha}{2\varepsilon N}\right)^{-N/2} = \mathbf{C}\nu^{k+1}.$$

Therefore we get

$$\|\mathbf{Y}_r^{[k+2]} - \mathbf{Y}_r^{[k+1]}\|_{\bar{\Omega}_r^N} \le C\nu^{k+1} \tag{5.13}$$

and consequently

$$\|\mathbf{Y}_r^{[k+2]} - \mathbf{Y}_r^{[k+1]}\|_{\bar{\Omega}_r^N \setminus \bar{\Omega}_c} \le C\nu^{k+1}. \tag{5.14}$$

Finally note that

$$\mathbf{L}^N(\mathbf{Y}_c^{[k+2]} - \mathbf{Y}_c^{[k+1]})(x_i) = \mathbf{0} \quad \text{for} \quad x_i \in \Omega_c^N, \quad (\mathbf{Y}_c^{[k+2]} - \mathbf{Y}_c^{[k+1]})(0) = \mathbf{0}.$$

Using our inductive hypothesis and (5.13), we have

$$|(\mathbf{Y}_c^{[k+2]} - \mathbf{Y}_c^{[k+1]})(1-\tau)| = |(\bar{\mathbf{Y}}_r^{[k+2]} - \bar{\mathbf{Y}}_r^{[k+1]})(1-\tau)|$$
$$= |(\mathbf{Y}_r^{[k+2]} - \mathbf{Y}_r^{[k+1]})(1-\tau)| \le C\nu^{[k+1]},$$

where we have used the fact that $(1-\tau)$ is the mesh point of $\bar{\Omega}_c^N$. Therefore, we can apply Lemma 4 to get

$$\|\mathbf{Y}_c^{[k+2]} - \mathbf{Y}_c^{[k+1]}\|_{\bar{\Omega}_c^N} \le C\nu^{k+1}. \tag{5.15}$$

Combining the estimates (5.14) and (5.15) we obtain,

$$\|\mathbf{Y}^{[k+2]} - \mathbf{Y}^{[k+1]}\|_{\bar{\Omega}^N} \le C\nu^{k+1}.$$

For $\tau = \frac{4\varepsilon}{\alpha} \ln N$ using the arguments given in Lemma 4.1 of [10] we obtain,

$$\nu = \left(1 + \frac{\tau\alpha}{2\varepsilon N}\right)^{-N/2} = \left(1 + \frac{2\ln N}{N}\right)^{-N/2} \le 2N^{-1}, \quad N \ge 1.$$

□

The following theorem is the main result of this paper, combining Lemmas 5 and 6 we prove that two iterations are sufficient to attain almost second-order convergence.

Theorem 2. *Let $\boldsymbol{y}(x)$ be the solution to (1.6)–(1.7) and $\mathbf{Y}^{[k]}(x_i)$ be the k^{th} iterate of the discrete Schwarz method described in Section 4. If $\tau = \frac{4\varepsilon}{\alpha} \ln N$ and $N > 2$, then*

$$\| \boldsymbol{Y}^{[k]} - \boldsymbol{y}\|_{\bar{\Omega}^N} \le CN^{-k} + CN^{-2} \ln^3 N.$$

Proof. From Lemma 6 there exists $\mathbf{Y}$ such that $\mathbf{Y} := \mathrm{lt}_{k\to\infty} \mathbf{Y}^{[k]}$. We know from Lemma 5 that there exists C such that

$$\|\mathbf{Y}^{[k]} - \boldsymbol{y}\|_{\bar{\Omega}^N} \le C2^{-k} + CN^{-2}\ln^3 N.$$

This implies that

$$\|\mathbf{Y} - \boldsymbol{y}\|_{\bar{\Omega}^N} \le CN^{-2}\ln^3 N. \tag{5.16}$$

Also from Lemma 6 that there exists C such that

$$\|\mathbf{Y}^{[k+1]} - \mathbf{Y}^{[k]}\|_{\bar{\Omega}^N} \le CN^{-k}.$$

Consequently, for $N \ge 2$, there exists C such that

$$\|\mathbf{Y}^{[k]} - \mathbf{Y}\|_{\bar{\Omega}^N} \le C\sum_{l=k}^{\infty} N^{-l} = C\left[\frac{N^{-k}}{1-N^{-1}}\right] \le CN^{-k}. \tag{5.17}$$

Thus, using (5.16) and (5.17), we conclude that

$$\begin{aligned}\|\mathbf{Y}^{[k]} - \boldsymbol{y}\|_{\bar{\Omega}^N} &= \|\mathbf{Y}^{[k]} - \mathbf{Y} + \mathbf{Y} - \boldsymbol{y}\|_{\bar{\Omega}^N} \\ &\le \|\mathbf{Y}^{[k]} - \mathbf{Y}\|_{\bar{\Omega}^N} + \|\mathbf{Y} - \boldsymbol{y}\|_{\bar{\Omega}^N} \le CN^{-k} + CN^{-2}\ln^3 N.\end{aligned}$$

□

6 Numerical experiments

In this section, we consider one example to illustrate the theoretical results for the BVPs (1.1)–(1.2). The stopping criterion for the iterative procedure is taken to be

$$\|\mathbf{Y}^{[k+1]} - \mathbf{Y}^{[k]}\|_{\bar{\Omega}^N} \le 10^{-14}, \text{ for } \quad j = 1, 2.$$

We normally omit the superscript k on the final Schwarz iterate and write simply Y_j^N. Let Y_j^N be a Schwarz numerical approximation for the exact solution y_j on the mesh Ω^N and N is the number of mesh points. For a finite set of values of $\varepsilon = \{2^0, \ldots, 2^{-30}\}$, we compute the maximum point-wise two mesh difference errors for $j = 1, 2$

$$\|Y_j^N - y_j\|_{\Omega^N} \approx D_{\varepsilon,j}^N := \|Y_j^N - \bar{Y}_j^{2N}\|_{\Omega^N}, \quad D_j^N = \max_{\varepsilon} D_{\varepsilon,j}^N,$$

where $\bar{Y}_j^{2N}$ is the numerical solution obtained on a mesh with the same transition points, but with $2N$ intervals in each subdomain. From these quantities the ε-uniform order of convergence is computed from

$$p_j^N = \log_2\left\{D_j^N / D_j^{2N}\right\}, \text{ for } \quad j = 1, 2.$$

The computed maximum pointwise errors $D_j^N, (j = 1, 2)$ and the computed order of convergence $p_j^N, (j = 1, 2)$ and k (the number of iterations computed) for various values of N and ε are tabulated in Table 1 and Table 2. The nodal errors are plotted as graphs in Figure 1. We can see that the errors decrease as N increases. The computed rates of convergence are almost second-order, with the usual $\ln N$ factor associated with these techniques.

Example 1. Consider the BVP

$$-\varepsilon y^{iv}(x) + (2-x)y'''(x) + (1+x)y''(x) - (x^2/5)y(x) = -\sinh x,$$
$$y(0)=0, \quad y''(0)=0, \quad y(1)=0, \quad y''(1)=0.$$

The numerical results are presented in Table 1 and Table 2.

Table 1. Values of D_1^N, p_1^N for the solution component Y_1 for the Example 1

	Number of mesh points N				
	64	128	256	512	1024
2^{0}	4.5356e-008	1.1447e-008	2.8752e-009	7.2046e-010	1.8032e-010
2^{-2}	1.7836e-007	4.4157e-008	1.0975e-008	2.7349e-009	6.8258e-010
2^{-4}	2.3850e-007	5.8368e-008	1.4400e-008	3.5733e-009	8.8980e-010
2^{-6}	5.5922e-007	7.3269e-008	1.0758e-008	2.6651e-009	6.6270e-010
2^{-8}	3.6738e-006	8.3723e-007	1.9091e-007	4.3482e-008	9.8857e-009
2^{-10}	5.1796e-006	1.2608e-006	3.0798e-007	7.5364e-008	1.8455e-008
2^{-12}	5.6146e-006	1.3864e-006	3.4362e-007	8.5332e-008	2.1212e-008
2^{-14}	5.7272e-006	1.4191e-006	3.5296e-007	8.7960e-008	2.1942e-008
2^{-16}	5.7556e-006	1.4274e-006	3.5532e-007	8.8625e-008	2.2128e-008
2^{-18}	5.7627e-006	1.4294e-006	3.5591e-007	8.8792e-008	2.2174e-008
2^{-20}	5.7645e-006	1.4299e-006	3.5606e-007	8.8834e-008	2.2186e-008
2^{-22}	5.7650e-006	1.4301e-006	3.5610e-007	8.8845e-008	2.2189e-008
2^{-24}	5.7651e-006	1.4301e-006	3.5610e-007	8.8847e-008	2.2189e-008
2^{-26}	5.7651e-006	1.4301e-006	3.5611e-007	8.8848e-008	2.2190e-008
2^{-28}	5.7651e-006	1.4301e-006	3.5611e-007	8.8848e-008	2.2190e-008
2^{-30}	5.7651e-006	1.4301e-006	3.5611e-007	8.8848e-008	2.2190e-008
D_1^N	5.7651e-006	1.4301e-006	3.5611e-007	8.8848e-008	2.2190e-008
p_1^N	**2.0112**	**2.0057**	**2.0029**	**2.0014**	-

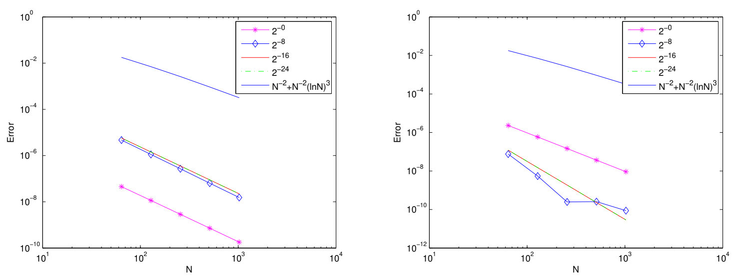

Figure 1. Nodal error for the components Y_1 and Y_2 of the Example 1

7 Conclusions

A singularly perturbed fourth-order ODEs of convection-diffusion problem is considered. It is shown that a designed discrete Schwarz method produces numerical approximations which converge in the maximum norm to the exact

Table 2. Values of D_2^N, p_2^N for the solution component Y_2 for the Example 1

	Number of mesh points N				
	64	128	256	512	1024
2^{0}	2.3114e-006	5.8302e-007	1.4641e-007	3.6683e-008	9.1808e-009
2^{-2}	6.1032e-006	1.5000e-006	3.7168e-007	9.2499e-008	2.3072e-008
2^{-4}	5.6156e-008	5.2490e-009	3.5293e-009	1.1476e-009	3.1933e-009
2^{-6}	6.5589e-008	1.9600e-008	5.0806e-009	1.2941e-009	3.2651e-010
2^{-8}	6.0556e-008	3.5318e-009	4.7276e-010	2.7546e-010	8.6772e-011
2^{-10}	1.0437e-007	1.1904e-008	1.2359e-009	9.5007e-011	2.5776e-012
2^{-12}	1.1620e-007	1.4187e-008	1.7060e-009	1.9786e-010	2.1020e-011
2^{-14}	1.1921e-007	1.4769e-008	1.8263e-009	2.2423e-010	2.7083e-011
2^{-16}	1.1997e-007	1.4916e-008	1.8565e-009	2.3086e-010	2.8610e-011
2^{-18}	1.2016e-007	1.4953e-008	1.8641e-009	2.3253e-010	2.8992e-011
2^{-20}	1.2021e-007	1.4962e-008	1.8660e-009	2.3294e-010	2.9088e-011
2^{-22}	1.2022e-007	1.4964e-008	1.8665e-009	2.3305e-010	2.9111e-011
2^{-24}	1.2022e-007	1.4965e-008	1.8666e-009	2.3307e-010	2.9117e-011
2^{-26}	1.2022e-007	1.4965e-008	1.8666e-009	2.3308e-010	2.9119e-011
2^{-28}	1.2022e-007	1.4965e-008	1.8666e-009	2.3308e-010	2.9119e-011
2^{-30}	1.2022e-007	1.4965e-008	1.8666e-009	2.3308e-010	2.9119e-011
D_2^N	6.1032e-006	1.5000e-006	3.7168e-007	9.2499e-008	2.3072e-008
p_2^N	**2.0246**	**2.0128**	**2.0066**	**2.0033**	-

solution. This convergence is shown to be of almost second-order. Note that from Theorem 2, for $k \geq 2$ the $N^{-2} + N^{-2} \ln^3 N$ term dominates the error bound. Thus, two iterations are sufficient to attained the desired accuracy.

The present method gives improved numerical results with regard to error and order compared with the other method in [2, 15, 16, 17]. From Theorem 2 it can be easily identified in which iterations, the Schwarz iterate terminates. From the given example number of iterations taken by this method is not more than two which is very much reduced when comparing iteration counts presented in [8, 9]. This illustrates the efficiency of the method used with proposed scheme in this paper.

Numerical experiment validate the theoretical result. The graphs plotted in the figure is convergent curves in the maximum norm at nodal points for the different values of ε and N for the example considered. This graph clearly indicate that the optimal error bound is of order $O(N^{-k} + N^{-2} \ln^3 N)$ as predicted.

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