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An Inexact Newton Method with Inner Preconditioned CG for Non-Uniformly Monotone Elliptic Problems

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Abstract. The present paper introduces an inexact Newton method, coupled with a preconditioned conjugate gradient method in inner iterations, for elliptic operators with non-uniformly monotone upper and lower bounds. Convergence is proved in Banach space level. The results cover real-life classes of elliptic problems. Numerical experiments reinforce the convergence results.

Keywords: inexact Newton iteration, conjugate gradients, nonlinear elliptic problems, iterative methods.

AMS Subject Classification: 65N30; 65J15; 47N20.

1 Introduction

Nonlinear elliptic problems have been the subject of extensive research in numerical analysis due to their applications in describing steady states in various fields of physics, such as elasticity, flow problems, nonlinear optics (see: [7,10,12,14]). In [16], nonlinear magnetic potential problems are discussed. Non-Newtonian fluids with power-law stess tensors are addressed in [15].

A widely used approach is to discretize the problem with finite elements (FEM), then apply e.g. a Newton-type iterative solver with conjugate gradient method used in inner iteration. The construction of such inner-outer iterations can be found in [11, 18], their framework for uniformly monotone elliptic problems has been presented in [1, 19], see also [17, 20] for recent applications. The present paper extends these methods for non-uniformly monotone elliptic operators.

The use of similar preconditioners for elliptic problems can be found in [13], where the authors introduce its applicability for quasi-Newton methods. This has also been extended recently for elliptic operators with non-uniformly monotone lower and upper bounds, see [8, 9].

This paper provides an inexact Newton method, coupled with preconditioned conjugate gradient method in inner iterations, for non-uniformly elliptic problems based on the setting of [8, 9, 13]. The preconditioners are based on spectrally equivalent operators. Additionally, the results of a numerical experiment for a subsonic flow model (see [5]) are provided as an example.

Section 2 contains the convergence result, Section 3 presents models that fall under our assumptions, while Section 4 shows results of the numerical experiment.

2 Abstract inner-outer iteration in Banach spaces

The theorems below show convergence results for inner-outer iteration in Banach space.

2.1 Convergence of the inexact Newton's method

We make the following assumptions.

ASSUMPTION 1. (i) Let X be a real Banach space with norm $\|\cdot\|$, and X' its dual, with usual notation $\langle v,\ u\rangle := vu$ (where $v \in X',\ u \in X$). The norm in X' is also denoted by $\|\cdot\|$.

(ii) We study operator equation

$$F(u) = 0, \tag{2.1}$$

where $F : X \to X'$ is a nonlinear operator with bihemicontinuous Gâteaux derivative. The latter is denoted by $F'(u)$ at given $u \in X$. The unique solution of equation (2.1) is denoted by u^*.

(iii) For any $u \in X$ the operator $F'(u)$ is symmetric.

(iv) There exists a continuous nonincreasing function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_0^{+\infty} \lambda(t)\, dt \;=\; +\infty$$

and

$$\langle F'(u)h, h\rangle \geq \lambda(\|u\|)\, \|h\|^2, \qquad \forall u, h \in X. \tag{2.2}$$

(v) There exists a continuous nondecreasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|F'(u) - F'(h)\| \leq L(\max\{\|u\|,\, \|h\|\})\, \|u - h\|, \qquad \forall u,\, h \in X.$$

Remark 1. If the function λ in (2.2) can be chosen constant, then the operator is uniformly monotone. However, we allow $\inf_{t\in\mathbb{R}^+} \lambda(t) = 0$, which means non-uniform monotonicity.

Algorithm 1. For arbitrary $u_0 \in X$ let $(u_n) \subset X$ be the sequence defined by

$$\begin{aligned} u_{n+1} &= u_n + p_n, && n \in \mathbb{N}, && (2.3)\\ \|F'(u_n)\, p_n + F(u_n)\|_n &\le \delta_n \|F(u_n)\|_n, && 0 < \delta_n \le \delta_0 < 1, && (2.4)\end{aligned}$$

where the energy norm $\|\cdot\|_n$ is defined below in (2.6) and

$$\exists c_\gamma > 0,\ 0 < \gamma \le 1, \text{ such that } \delta_n \le c_\gamma \|F(u_n)\|_n^\gamma. \tag{2.5}$$

Theorem 1. *Let Assumption 1 (i)–(v) be satisfied. Then the sequence defined by Algorithm 1 converges locally to u^* with order $(1+\gamma)$, namely, there exists a neighbourhood U of u^* that for a given $u_0 \in U$ there exists constants $C > 0$ and $0 < Q < 1$ such that*

$$\|u_n - u^*\| \le C Q^{(1+\gamma)^n}, \quad n \in \mathbb{N}.$$

Some lemmas and definitions in [9] that are needed for the proof are repeated below, mainly for the sake of convenience. The proofs shown in the reference for the lemmas apply here, if not shown otherwise below.

Lemma 1. *Equation (2.1) has a unique solution $u^* \in X$.*

We define the following energy norms in X':

$$\begin{gathered} \|v\|_u := \langle v,\, F'(u)^{-1} v\rangle^{1/2} \quad (\text{for given } u \in X),\\ \|\cdot\|_* := \|\cdot\|_{u^*}, \quad \|\cdot\|_n := \|\cdot\|_{u_n} \quad (\text{for given } n \in \mathbb{N}), \end{gathered} \tag{2.6}$$

and strictly increasing function $\Lambda:\ \mathbb{R}^+ \to \mathbb{R}^+,\ t \mapsto L(t)t + \|F'(0)\|$. For fixed $u \in X$, the norms $\|\cdot\|_u$ and $\|\cdot\|$ are equivalent, namely:

Lemma 2. *Denoting $n(u) := \frac{\lambda(\|u\|)}{\Lambda^{1/2}(\|u\|)}$, $N(u) := \frac{\Lambda(\|u\|)}{\lambda^{1/2}(\|u\|)}$, we have*

$$n(u)\|v\|_u \le \|v\| \le N(u)\|v\|_u, \qquad \forall v \in X'.$$

Specifically:

$$\begin{gathered} \tilde{\lambda}_*^{1/2}\|v\|_* \le \|v\| \le \tilde{\Lambda}_*^{1/2}\|v\|_*, \quad \forall v \in X',\\ \text{where } \tilde{\lambda}_* := \frac{\lambda^2(\|u^*\|)}{\Lambda(\|u^*\|)}, \quad \tilde{\Lambda}_* := \frac{\Lambda^2(\|u^*\|)}{\lambda(\|u^*\|)} \end{gathered} \tag{2.7}$$

and

$$\begin{gathered} \tilde{\lambda}_n^{1/2}\|v\|_n \le \|v\| \le \tilde{\Lambda}_n^{1/2}\|v\|_n, \quad \forall v \in X',\\ \text{where } \tilde{\lambda}_n := \frac{\lambda^2(\|u_n\|)}{\Lambda(\|u_n\|)}, \quad \tilde{\Lambda}_n := \frac{\Lambda^2(\|u_n\|)}{\lambda(\|u_n\|)}. \end{gathered} \tag{2.8}$$

Lemma 3. *There exists a strictly increasing function $R^* : \mathbb{R}^+ \to \mathbb{R}^+$, such that*

$$\frac{1}{1 + R^*(\|F(u_n)\|_*)} \le \frac{\|v\|_*^2}{\|v\|_n^2} \le 1 + R^*(\|F(u_n)\|_*), \qquad v \in X'.$$

The investigation of norms of elements of X in certain segments leads to the following observation.

Lemma 4. *There exists a nonincreasing function $\lambda_* : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\lambda(\|u\|) \geq \lambda_*(\|F(u)\|_*), \quad u \in X. \tag{2.9}$$

Lemma 5. *The following inequality holds*

$$\|F'(u_n)^{-1}\| \leq \frac{1}{\lambda_*(\|F(u_n)\|_*)}. \tag{2.10}$$

Proof. (2.2) entails:

$$\lambda(\|u_n\|)\|h\|^2 \leq \langle F'(u_n)h, h\rangle \leq \|F'(u_n)h\|\|h\|, \quad \forall h \in X$$

owing to $F'(u_n)$ being a bijection, by (2.9), we have (2.10). □

Lemma 6. *There exists a strictly increasing function $\Phi^* : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\tilde{L}_{n,n+1} := L(\max\{\|u_n\|, \|u_{n+1}\|\}) \leq L(\Phi^*(\|F(u_n)\|_*)). \tag{2.11}$$

Proof. The following result can be readily obtained from [9]. There exists a strictly increasing function $G^* : \mathbb{R}^+ \to \mathbb{R}^+$, such that

$$\|u\| \leq G^*(\|F(u)\|_*), \qquad u \in X. \tag{2.12}$$

Let us define $p_n^* := -F'(u_n)^{-1}F(u_n)$, which is the Newton step, and write expansion

$$u_{n+1} = (u_n + p_n^*) + (p_n - p_n^*), \tag{2.13}$$

where, due to (2.7), (2.10) and (2.12), the following estimation holds for the first term

$$\|u_n + p_n^*\| \leq \|u_n\| + \|F'(u_n)^{-1}\|\|F(u_n)\| \leq G^*(\|F(u_n)\|_*) + \frac{\tilde{\Lambda}_*^{1/2}}{\lambda_*(\|F(u_n)\|_*)}\|F(u_n)\|_*.$$

On the other hand, one can write

$$p_n - p_n^* = F'(u_n)^{-1}F'(u_n)(p_n - p_n^*),$$

using (2.4), (2.5), (2.8), Lemma 3 and (2.10) results in the following estimation of the second term of (2.13)

$$\begin{aligned}\|p_n - p_n^*\| =& \|F'(u_n)^{-1}\|\|F'(u_n)(p_n - p_n^*)\| \\ \leq& \frac{\tilde{\Lambda}_n^{1/2}}{\lambda_*(\|F(u_n)\|_*)} c_\gamma (1 + R^*(\|F(u_n)\|_*))^{\frac{1+\gamma}{2}} \|F(u_n)\|_*^{1+\gamma},\end{aligned}$$

the result follows. □

Lemma 7. *The following estimate holds for all $u, v \in X$:*

$$\|F(u) - F(v)\| \geq \lambda(\max\{\|u\|, \|v\|\})\|u - v\|,$$

in particular:

$$\|u_n - u^*\| \leq \|F(u_n)\|/\lambda(\max\{\|u_n\|, \|u^*\|\}). \tag{2.14}$$

Proof of Theorem 1. For given $n \in \mathbb{N}$, one can write expansion

$$F(u_{n+1}) = F(u_n) + F'(u_n)(u_{n+1} - u_n) + R(u_n),$$
$$\text{where } \|R(u_n)\| \leq \frac{\tilde{L}_{n,n+1}}{2}\|u_{n+1} - u_n\|^2,$$

by (2.3) and (2.4) we obtain

$$\|F(u_{n+1})\|_n \leq \delta_n \|F(u_n)\|_n + \|R(u_n)\|_n,$$

applying (2.8) and (2.11) entail

$$\|F(u_{n+1})\|_n \leq \delta_n \|F(u_n)\|_n + \frac{L(\Phi^*(\|F(u_n)\|_*))}{2\tilde{\lambda}_n^{1/2}}\|p_n\|^2. \tag{2.15}$$

Here, (2.8), (2.4), Lemma 5 and Lemma 3 imply

$$\begin{aligned}
\|p_n\| \leq & \tilde{\Lambda}_n^{1/2}\|F'(u_n)^{-1}\|\|F'(u_n)p_n\|_n \\
\leq & \tilde{\Lambda}_n^{1/2}\|F'(u_n)^{-1}\|(\|F(u_n) + F'(u_n)p_n\|_n + \|F(u_n)\|_n) \\
\leq & \frac{\tilde{\Lambda}_n^{1/2}}{\lambda_*(\|F(u_n)\|_*)}\|F(u_n)\|_n(1 + \delta_n) \\
\leq & \frac{\tilde{\Lambda}_n^{1/2}}{\lambda_*(\|F(u_n)\|_*)}(1 + \delta_n)(1 + R^*(\|F(u_n)\|_*))^{1/2}\|F(u_n)\|_*.
\end{aligned}$$

Combining this and (2.15), then using (2.5), and applying Lemma 3 again

$$\begin{aligned}
&\|F(u_{n+1})\|_* \leq (1 + R^*(\|F(u_n)\|_*))^{\frac{3}{2}+\gamma}\Big(c_\gamma \|F(u_n)\|_*^{1+\gamma} + \\
&+ \frac{L(\Phi^*(\|F(u_n)\|_*))\tilde{\Lambda}_n}{2\tilde{\lambda}_n^{1/2}\lambda_*^2(\|F(u_n)\|_*)}(1 + c_\gamma\|F(u_n)\|_*^\gamma)^2\|F(u_n)\|_*^2\Big).
\end{aligned} \tag{2.16}$$

To conclude local convergence by induction, we need a slightly different approach compared to [13] and [9], due to inner iteration. From (2.16), we obtain

$$\|F(u_{n+1})\|_* \leq \varphi(\|F(u_n)\|_*)\|F(u_n)\|_*^{1+\gamma}, \tag{2.17}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing function

$$\varphi(\|F(u_n)\|_*) := a(\|F(u_n)\|_*)\Big(c_\gamma + b(\|F(u_n)\|_*)(1 + c_\gamma\|F(u_n)\|_*^\gamma)^2\|F(u_n)\|_*^{1-\gamma}\Big)$$

with strictly increasing and nondecreasing functions $a,\ b : \mathbb{R}^+ \to \mathbb{R}^+$, respectively:

$$a(t) := (1 + R^*(t))^{\frac{3}{2}+\gamma}, \quad b(t) := \frac{L(\Phi^*(t))\tilde{\Lambda}_n}{2\tilde{\lambda}_n^{1/2}\lambda_*^2(t)}.$$

Similarly, we can construct a function $\varphi_1 : \mathbb{R}^+ \to \mathbb{R}^+$, that is also strictly increasing, and

$$\|F(u_{n+1})\|_* \le \varphi_1(\|F(u_n)\|_*)\|F(u_n)\|_*$$

can be written, namely: $\varphi_1(t) := \varphi(t)t^\gamma$.

If $\varphi_1(\|F(u_0)\|_*) < 1$, then, due to φ_1 being increasing, $\|F(u_n)\|_* < \|F(u_0)\|_*$ for all $n \in \mathbb{N}$ by induction. Let $r := \varphi(\|F(u_0)\|_*)$, then (2.17) yields by induction that

$$\begin{gathered}\|F(u_n)\|_* \le r^{\frac{(1+\gamma)^n-1}{\gamma}}\|F(u_0)\|_*^{(1+\gamma)^n} \le dQ^{(1+\gamma)^n}, \\ \text{where } d := r^{-1/\gamma},\ Q := r^{1/\gamma}\|F(u_0)\|_* = \varphi_1^{1/\gamma}(\|F(u_0)\|_*) < 1,\end{gathered} \tag{2.18}$$

thus $\lim_{n\to\infty} \|F(u_n)\|_* = 0$, and $\lim_{n\to\infty} \varphi_1(\|F(u_n)\|_*) = 0$.

Using (2.9) on (2.14), then (2.7) gives

$$\|u_n - u^*\| \le \tilde{\Lambda}_*^{1/2}\|F(u_n)\|_*/\lambda_*(\|F(u_n)\|_*).$$

Combining this with (2.18), and using definition $C := d\tilde{\Lambda}_*^{1/2}/\lambda_*(\|F(u_0)\|_*)$, we get:

$$\|u_n - u^*\| \le CQ^{(1+\gamma)^n}.$$

□

Remark 2. One can obtain, using [9], the following inequalities for the convergence for $\gamma < 1$:

$$\limsup \frac{\|F(u_{n+1})\|_*}{\|F(u_n)\|_*} = 0, \quad \limsup \frac{\|F(u_{n+1})\|_*}{\|F(u_n)\|_*^{1+\gamma}} \le c_\gamma.$$

2.2 Inner-outer iteration

In what follows, the applied inner iteration is specified, i. e., for given $n \in \mathbb{N}$, the method of obtaining approximate solution $p_n \in X$ to auxiliary equation

$$F'(u_n)p_n^* = -F(u_n).$$

Let us introduce the energy inner product on X as $\langle x, y\rangle_B = \langle Bx, y\rangle$.

Algorithm 2. For fixed $n \in \mathbb{N}$, since $F'(u_n)$ is a uniformly positive bounded linear symmetric operator, we can apply the preconditioned conjugate gradient method to obtain the p_n, namely, let B_n be a uniformly positive bounded linear symmetric operator, for which

$$m_n\langle B_n h, h\rangle \le \langle F'(u_n)h, h\rangle \le M_n\langle B_n h, h\rangle, \quad \forall h \in X \tag{2.19}$$

holds ($M_n \ge m_n > 0$). The resulting sequence is $\left(p_n^{(k)}\right) \subset X$, where $k \in \mathbb{N}$ is the index corresponding to the inner iteration, and we set $\left(p_n^{(0)}\right) := 0$.

Furthermore, let us denote the error and the residual error $e_n^{(k)} := p_n^{(k)} - p_n^*$, and $r_n^{(k)} := F'(u_n)e_n^{(k)} = F'(u_n)p_n^{(k)} + F(u_n)$, respectively. The step is defined as follows, where s_n denotes the conjugate directions:

$$p_n^{(k+1)} := p_n^{(k)} + \alpha_n^{(k)} s_n^{(k)}, \quad r_n^{(k+1)} := r_n^{(k)} + \alpha_n^{(k)} z_n^{(k)}, \quad \text{where :}$$

$$B_n z_n^{(k)} := F'(u_n)s_n^{(k)}, \quad \text{and } \alpha_n^{(k)} := -\frac{\|r_n^{(k)}\|_{B_n}^2}{\langle F'(u_n)s_n^{(k)}, s_n^{(k)}\rangle},$$

$$s_n^{(k+1)} := r_n^{(k+1)} + \beta_n^{(k)} s_n^{(k)}, \quad \text{where : } \beta_n^{(k)} := \frac{\|r_n^{(k+1)}\|_{B_n}^2}{\|r_n^{(k)}\|_{B_n}^2}.$$

By (2.19), we have

$$m_n\|h\|_{B_n}^2 \le \langle B_n^{-1}F'(u_n)h, h\rangle_{B_n} \le M_n\|h\|_{B_n}^2, \quad \forall h \in X,$$

therefore, the conjugate gradient method can be applied [3] in the energy space corresponding to operator B_n.

We choose outer iteration step

$$p_n := p_n^{(k)}, \quad \text{for some } k \ge k_{n,min}, \quad \text{where } k_{n,min} = \left\lceil \frac{\ln(\delta_n/2)}{\ln(Q_n)} \right\rceil \tag{2.20}$$

is the minimum number of iterations, and

$$Q_n := \frac{\sqrt{M_n} - \sqrt{m_n}}{\sqrt{M_n} + \sqrt{m_n}}. \tag{2.21}$$

Theorem 2. *Let Assumptions 1 be satisfied. For the sequence generated by Algorithm 1, let us apply Algorithm 2 in the inner iteration in each step. If the two iterations are connected by (2.20)–(2.21), then we have*

$$\|F'(u_n)p_n^{(k)} + F(u_n)\|_n \le 2Q_n^k\|F(u_n)\|_n, \quad n, k \in \mathbb{N} \tag{2.22}$$

and (2.4) holds for all $n \in \mathbb{N}$.

Proof.

For the conjugate gradient method in the energy space corresponding to operator B_n, the following is known

$$\frac{\|e_n^{(k)}\|_{F'(u_n)}}{\|e_n^{(0)}\|_{F'(u_n)}} \le 2Q_n^k,$$

on the other hand

$$\|e_n^{(k)}\|_{F'(u_n)}^2 = \langle F'(u_n)e_n^{(k)}, e_n^{(k)}\rangle = \langle r_n^{(k)}, F'(u_n)^{-1}r_n^{(k)}\rangle = \|r_n^{(k)}\|_{F'(u_n)^{-1}}^2.$$

Combining these and using $\|r_n^{(k)}\|_{F'(u_n)^{-1}} = \|r_n^{(k)}\|_n$ yields

$$\|r_n^{(k)}\|_n \le 2Q_n^k\|r_n^{(0)}\|_n,$$

since $r_n^{(0)} = F(u_n)$, (2.22) follows.

Therefore, inequality

$$2Q_n^k \le \delta_n$$

is sufficient for (2.4) to hold, and this follows from assumption (2.20). □

3 Elliptic models

By [9], the following boundary value problems posed in $W^{1,p}(\Omega)$ fall under the assumptions above. Such BVPs arise e.g. in non-Newtonian fluids [15], bending of elastic beams [6], etc.

Firstly, as a general nonlinearity, let us consider the class of models described by

$$\begin{cases} -\text{div } f(x, \nabla u) &= \omega, \\ u_{|\partial\Omega} &= 0, \end{cases} \tag{3.1}$$

where $f: \ \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 nonlinear vector field. It is assumed to have symmetric Jacobians $\frac{\partial f(x,\, \eta)}{\partial \eta}$, that satisfy:

$$c_1 \left(k_0 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi|^2 \le \frac{\partial f}{\partial \eta}(x,\eta)\, \xi \cdot \xi \le \widetilde{c}_1 \left(k_0 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi|^2, \tag{3.2}$$

$$\left\| \frac{\partial f}{\partial \eta}(x,\eta_1) - \frac{\partial f}{\partial \eta}(x,\eta_2) \right\| \le d_1 \max_{\eta \in [\eta_1,\eta_2]} \left\{ \left(k_0 + |\eta|^2\right)^{\frac{p-3}{2}} \right\} |\eta_1 - \eta_2|, \tag{3.3}$$

$\forall x \in \Omega, \ \ \xi, \eta, \eta_1, \eta_2 \in \mathbb{R}^n$ for some constants $1 < p < \infty, \quad \widetilde{c}_1 \ge c_1 > 0, \quad k_0 > 0$, and we assume $\omega \in L^{p'}(\Omega)$.

One can also use mixed boundary conditions in (3.1)

$$u_{\,|\Gamma_D} = 0, \qquad f(x,\, \nabla u) \cdot \mathbf{n}_{\,|\Gamma_N} = \gamma, \tag{3.4}$$

where $\Gamma_D \cup \Gamma_N = \partial\Omega$. Then the solution is looked for in the subspace $\{u \in W^{1,p}(\Omega) : \ u_{\,|\Gamma_D} = 0\}$.

In particular, we may have a given scalar nonlinearity:

$$f(x,\, \eta) := a(x,\, |\eta|^2)\, \eta \qquad \text{for } (x,\, \eta) \in \Omega \times \mathbb{R}^n, \tag{3.5}$$

where $a: \ \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 scalar-valued function. Then problem (3.1) with mixed boundary conditions (3.4) becomes

$$\begin{cases} -\text{div}\big(a(x,\, |\nabla u|^2)\; \nabla u\big) &= \omega \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \Gamma_D\,, \\ a(x, |\nabla u|^2)\, \frac{\partial u}{\partial \mathbf{n}} &= \gamma \quad \text{on } \Gamma_N\,. \end{cases} \tag{3.6}$$

If we assume that for all $(x,\ r) \in \Omega \times \mathbb{R}^+$, $a(x, r^2) \in C^2$ w.r.t. r, furthermore

$$\underline{a}(x, r^2) := \min\left\{ a(x, r^2),\ \frac{\partial}{\partial r}\big(a(x, r^2)\, r\big) \right\} \ge c_1 \left(k_0 + r^2\right)^{\frac{p-2}{2}} \tag{3.7}$$

$$\text{and} \quad \frac{\partial a}{\partial r}(x, r^2) \le d_2 (k_0 + r^2)^{\frac{p-4}{2}}, \qquad \frac{\partial^2 a}{\partial r^2}(x, r^2) \le d_3 (k_0 + r^2)^{\frac{p-6}{2}}, \tag{3.8}$$

then the vector field (3.5) satisfies (3.2)–(3.3).

In accordance with [9], the definition of operators B_n can be given with scalar coefficients

$$\langle B_n h, v \rangle = \int_\Omega \beta(x,\, |\nabla u_n|^2)\; \nabla h \cdot \nabla v, \qquad \forall h, v \in V_h.$$

Let us define $\overline{a}$ by replacing *min* with *max* in (3.7). If the function β satisfies

$$\underline{a}(x,r^2) \le \beta(x,\, r^2) \le \overline{a}(x,r^2), \quad \text{e.g.} \quad \beta(x,\, r^2) := \frac{1}{2}\left(\underline{a}(x,\, r^2) + \overline{a}(x,\, r^2)\right),$$

then, for all n, spectral equivalence of B_n and $F'(u_n)$ can be obtained readily, in other words, (2.19) holds.

The Newton equation must be discretized which adds to the inexactness. This can be based on a combination of a coarse and a fine mesh which can save computer time (see [2,4,21], etc.). Future work in the topic might include corresponding investigations.

4 Numerical experiment

4.1 Subsonic flow example

The following boundary value problem describing potential flow in a wind tunnel section $\Omega \subset \mathbb{R}^2$ has been presented in [5, Chap. 9]. For the geometry, see [5, Fig. 5]. Let us consider:

$$\begin{cases} -\text{div}\,\left(\varrho(|\nabla u|^2)\,\nabla u\right) &= & 0 & \text{in } \Omega\,, \\ \varrho(|\nabla u|^2)\,\frac{\partial u}{\partial \mathbf{n}} &= & \gamma & \text{on } \Gamma_N\,, \\ u &= & v_\infty & \text{on } \Gamma_D\,, \end{cases} \tag{4.1}$$

the scalar nonlinearity is $\varrho(|\nabla u|^2) = \varrho_\infty\left(1+\frac{1}{5}(M_\infty^2-|\nabla u|^2)\right)^{5/2}$, where $\varrho_\infty > 0$ is the air density at infinity, and u is the velocity potential. $M_\infty > 0$ denotes the Mach number at infinity, v_∞ stands for the constant velocity potential on Dirichlet boundary Γ_D, while $\Gamma_N := \partial\Omega \setminus \Gamma_D$ is the Neumann boundary. The range of γ is $\{0,\ \tilde{v}_\infty\}$, where $\tilde{v}_\infty > 0$ is a parameter describing outflow velocity.

We only deal with the case $v_\infty = 0$ without the loss of generality, since u is a potential (one may observe that (4.1) only contains derivatives of u except for the constant Dirichlet boundary condition).

By involving problem (4.1), our goal is to test that our method may work even beyond the limitations posed by our previous theoretical assumptions. Namely, the condition for ellipticity is that $|\nabla u|$ is pointwise below the subsonic limit, hence the operator cannot be defined on a whole function space. Therefore, the above subsonic flow problem described by (4.1) is not precisely contained in the elliptic model (3.6) with assumptions (3.7)–(3.8). However, one can expect that the method of present paper converges properly while the solution and the utilized part of the iterative sequence satisfy the subsonic limit condition.

We apply the results of Section 2, and use the finite element method (FEM) for the discretization of the problem, namely, Courant elements. Hence the above Banach space X is a finite dimensional space consisting of Courant elements for which $u_{|\Gamma_D} = 0$. Therefore, all norms are equivalent.

Thus one can define the operator describing (4.1) in weak form as

$$\langle F(u),\ v\rangle \equiv \int_\Omega \rho(|\nabla u|^2)\nabla u\cdot\nabla v - \int_{\Gamma_N}\gamma v, \quad \forall u,\ v\in V_h.$$

Consequently, the FEM problem becomes the task of finding $u \in V_h$, such that $\langle F(u),\ v\rangle = 0,\ \forall v \in V_h$. This can be written shortly in the form of (2.1) as

$$F(u) = 0 \quad \text{in } V_h.$$

The Gâteaux derivative of the operator can be obtained readily in weak form

$$\langle F'(u)h,\ v\rangle = \int_\Omega \left(\rho(|\nabla u|^2)\nabla h \cdot \nabla v + 2\rho'(|\nabla u|^2)(\nabla u \cdot \nabla h)(\nabla u \cdot \nabla v)\right), \quad \forall h,\ v \in V_h.$$

The applied preconditioner in the n-th outer step is

$$\langle B_n h,\ v\rangle = \int_\Omega \left(\rho(|\nabla u_n|^2) + \rho'(|\nabla u_n|^2)|\nabla u_n|^2\right) \nabla h \cdot \nabla v, \quad \forall h,\ v \in V_h.$$

It provides a substantial simplification of the Gâteaux derivative of the operator.

4.2 Numerical results

The results of the above experiment with five different meshes and two different $\tilde{v}_\infty$ values are presented below.

Let symbol DoF stand for the degrees of freedom of the FEM model. Denote $n_1,\ n_2$ the number of outer iteration steps necessary to achieve smaller relative residual error than 10^{-4} and 10^{-6}, respectively. Let k denote the number of inner iteration steps required for relative residual error to be smaller than 10^{-4} for an individual outer iteration step. The numerical results are summarized in Table 1, where the outer step number, which the given value of k corresponds to, can be identified in the header.

As a conclusion, we state the following observations. Firstly, we readily find robustness of the method developed here for the subsonic flow example. A similar robustness result for a quasi-Newton method with the same preconditioner can be found in [9].

Secondly, (2.20) states for the subsonic model that at most 8 inner iterations are sufficient to reach relative tolerance 10^{-4}. Comparing this to Table 1 shows that in fact far less iterations can be sufficient as well.

Table 1. The number of required outer (n_1, n_2) and inner (k) iteration steps

DoF	$\tilde{v}_\infty = 0.4$						$\tilde{v}_\infty = 0.6$						
	n_1	n_2	k				n_1	n_2	k				
			1	2	3	4			1	2	3	4	5
243	3	4	1	3	3	3	4	5	1	3	4	4	4
884	3	4	1	3	3	3	4	5	1	3	3	4	4
3432	3	4	1	2	3	3	4	5	1	3	4	4	4
13520	3	4	1	2	3	3	4	5	1	3	4	4	4
53664	3	4	1	2	3	3	4	5	1	3	4	4	4

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