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A Degenerating Robin-Type Traction Problem in a Periodic Domain

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Abstract. We consider a linearly elastic material with a periodic set of voids. On the boundaries of the voids we set a Robin-type traction condition. Then, we investigate the asymptotic behavior of the displacement solution as the Robin condition turns into a pure traction one. To wit, there will be a matrix function $b[k](\cdot)$ that depends analytically on a real parameter k and vanishes for $k = 0$ and we multiply the Dirichlet-like part of the Robin condition by $b[k](\cdot)$. We show that the displacement solution can be written in terms of power series of k that converge for k in a whole neighborhood of 0. For our analysis we use the Functional Analytic Approach.

Keywords: Robin boundary value problem, integral representations, integral operators, integral equations methods, linearized elastostatics, periodic domain.

AMS Subject Classification: 35J65; 31B10; 45F15; 74B05.

1 Introduction

There is almost a century of literature on the mathematical analysis of perforated plates and porous materials (see, e.g., [4,15,19,20,30], see also Mityushev *et al.* [2] for a review). Examples can be found in the applications to the study of porous coatings and of interfacial coatings. If the thickness of the coating is smaller than the characteristic size of the pores (while its material properties are weaker/softer in comparison with those of the main composite material), the corresponding mathematical problems may degenerate into periodic problems with conditions of Robin type (see [3, 22, 25, 26, 31, 32], see also the books by Milton [24, Chap. 1] and Movchan *et al.* [28]). In dimension two, these problems can be analyzed with complex variable techniques (see, e.g., Kapanadze *et al.* [21], Drygaś *et al.* [11], Gluzman *et al.* [14]). For analog problems in dimension $n \geq 2$, one may resort to integral equation methods (as, e.g., in Ammari and Kang [1]). Perturbed boundary conditions in linear elasticity have been analyzed in the framework of Homogenization Theory, for example in Gómez, Nazarov and Pérez-Martínez [16, 17].

In this paper, we study the Lamé system in a periodic domain with a Robin-type traction condition on the boundary. The trace of the displacement part (we may also say the "Dirichlet-type term") of the boundary condition is multiplied by a matrix function $b[k](\cdot)$ that depends analytically on a positive parameter k and that vanishes for $k = 0$. Then, as k approaches 0 the Robin boundary condition degenerates into a pure traction one (a natural condition to have when dealing with the Lamé equations). We study the map that takes k to the displacement solution $u[k]$ and, under suitable conditions on $b[k](\cdot)$, we show that $k \mapsto u[k]$ can be described in terms of power series of k that converge for k in a neighborhood of 0. A similar result was obtained in [29] for the analog problem in a bounded domain with a single hole. Here we show that the approach of [29] can be adapted to the case of infinite periodic domains.

1.1 The problem

We start by presenting the geometric setting. We fix once for all

$$n \in \mathbb{N} \setminus \{0, 1\}, \qquad (q_{11}, \ldots, q_{nn}) \in]0, +\infty[^n .$$

Here, $\mathbb{N}$ denotes the set of natural numbers including 0. We take

$$Q := \Pi_{j=1}^{n}]0, q_{jj}[$$

as fundamental periodicity cell and we denote by q the diagonal matrix with (j, j) entry equal to q_{jj} for all $j \in \{1, \ldots, n\}$. We construct our periodic domain by removing from $\mathbb{R}^n$ congruent copies of a bounded domain of class $C^{m,\alpha}$. (For the definition of sets and functions of the Schauder class $C^{j,\alpha}$ ($j \in \mathbb{N}$) we refer, e.g., to Gilbarg and Trudinger [13]). Therefore, we fix once for all a natural number $m \in \mathbb{N} \setminus \{0\}$, a real number $\alpha \in]0, 1[$, and we assume that

$$\Omega_Q \text{ is a bounded open subset of } \mathbb{R}^n \text{ of class } C^{m,\alpha} \text{ such that } \overline{\Omega_Q} \subseteq Q.$$

We define the periodic domain (see Figure 1)

$$\mathbb{S}[\Omega_Q]^- := \mathbb{R}^n \setminus \bigcup_{z \in \mathbb{Z}^n} (qz + \overline{\Omega_Q}).$$

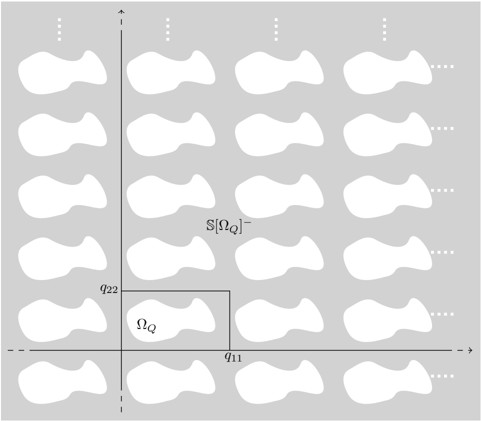

Figure 1. A 2-dimensional example of the periodically perforated set $\mathbb{S}[\Omega_Q]^-$.

To introduce the Lamé equations in $\mathbb{S}[\Omega_Q]^-$, we denote by T the function from $]1-(2/n),+\infty[\times M_n(\mathbb{R})$ to $M_n(\mathbb{R})$ defined by

$$T(\omega, A):=(\omega-1)(\operatorname{tr}A)I_n + (A + A^t)$$

for all $\omega \in]1-(2/n),+\infty[$, $A \in M_n(\mathbb{R})$. Here, $M_n(\mathbb{R})$ denotes the space of $n \times n$ matrices with real entries, I_n denotes the $n \times n$ identity matrix, $\operatorname{tr}A$ and A^t denote the trace and the transpose matrix of A, respectively. We note that if we set $L[\omega]:=\Delta + \omega\nabla\operatorname{div}$, then, $L[\omega]u = \operatorname{div} T(\omega, Du)$ for all regular vector valued functions u, where Du denotes the Jacobian matrix of u.

Then, we take a matrix $B \in M_n(\mathbb{R})$, a function $g \in C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)$, and a function $b \in C^{m-1,\alpha}(\partial\Omega_Q, M_n(\mathbb{R}))$, such that

- $\xi^t b(x)\xi \leq 0$ for all $\xi \in \mathbb{R}^n$ and all $x \in \partial\Omega_Q$,
- $\det \int_{\partial\Omega_Q} b\, d\sigma \neq 0$.

(Note that the last condition implies that $\det b(x_0) \neq 0$ for some $x_0 \in \partial\Omega_Q$.) With these ingredients we write a boundary value problem for the Lamé equation in $\mathbb{S}[\Omega_Q]^-$ with a Robin-type boundary condition. That is,

$$\begin{cases} \operatorname{div} T(\omega, Du) = 0 \ \text{ in } \mathbb{S}[\Omega_Q]^- , \\ u(x + qe_j) = u(x) + Be_j \ \ \forall x \in \overline{\mathbb{S}[\Omega_Q]^-}, \ \ \forall j \in \{1, \dots, n\}, \\ T(\omega, Du(x))\nu_{\Omega_Q}(x) + b(x)u(x) = g(x) \ \ \forall x \in \partial\Omega_Q , \end{cases} \tag{1.1}$$

where $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$ and ν_{Ω_Q} denotes the outward unit normal to $\partial\Omega_Q$. The matrix B determines a variation of the displacement function on the opposite sides of the fundamental cell. In particular, the displacement is periodic when $B = 0$. From the physics view point, a matrix B different from zero can result from the application of an external load that affects the entire matrix material. Just for comparison, when dealing with similar problem for the Laplace equation, $B \neq 0$ signals the presence of a heat flow through the enveloping matrix (cf. [9], Mityushev, Pesetskaya and Rogosin [27]).

It is well known that, for $\omega \in]1-(2/n), +\infty[$, the solution u of (1.1) exists, is unique, and belongs to the Schauder class $C^{m,\alpha}(\overline{\mathbb{S}[\Omega_Q]^-}, \mathbb{R}^n)$. A proof can be found, for example, in [8, Thm. 4.4] and the argument follows a quite standard strategy: first, we transform the problem into a periodic one and we prove the uniqueness of the solution by an energy estimate. Then, we use a periodic analog of the elastic single layer potential to transform the periodic problem into a Fredholm integral equation of the second kind, and finally, we obtain the existence and the regularity of the solution by the properties of the single layer potential.

We now want the term $b(x)u(x)$ in the boundary condition of (1.1) to disappear in a suitable way as a certain positive parameter k tends to zero. Then, we take $k_0 > 0$ and we introduce an analytic map

$$]-k_0, k_0[\ni k \mapsto b[k] \in C^{m-1,\alpha}(\partial\Omega_Q, M_n(\mathbb{R}))$$

that satisfies the following conditions:

- $\xi^t b[k](x)\xi \leq 0$ for all $\xi \in \mathbb{R}^n$, all $x \in \partial\Omega_Q$, and all $k \in]0, k_0[$, (1.2)
- $\det \int_{\partial\Omega_Q} b[k]\, d\sigma \neq 0$ for all $k \in]0, k_0[$, (1.3)
- $b[0] = \lim_{k\to 0} b[k] = 0$, (1.4)

and for a fixed $\omega \in]1-(2/n), +\infty[$ we consider the following problem

$$\begin{cases} \operatorname{div} T(\omega, Du) = 0 \ \text{ in } \mathbb{S}[\Omega_Q]^-, \\ u(x+qe_j) = u(x) + Be_j \ \ \forall x \in \overline{\mathbb{S}[\Omega_Q]^-}, \ \ \forall j \in \{1,\dots,n\}, \\ T(\omega, Du(x))\nu_{\Omega_Q}(x) + b[k](x)u(x) = g(x) \ \ \forall x \in \partial\Omega_Q\,. \end{cases} \tag{1.5}$$

For $k \in]0, k_0[$ problem (1.5) is a problem of the same kind as (1.1). Thus, for each $k \in]0, k_0[$, problem (1.5) has a unique solution $u \in C^{m,\alpha}(\overline{\mathbb{S}[\Omega_Q]^-}, \mathbb{R}^n)$, and we denote it by $u[k]$ to emphasize its dependence on k. On the other hand, for $k = 0$ we have $b[0] = 0$ and the Robin-type traction condition of problem (1.5) turns into a Neumann-type one. To wit, when $k = 0$ the boundary condition of problem (1.5) is

$$T(\omega, Du(x))\nu_{\Omega_Q}(x) = g(x) \qquad \forall x \in \partial\Omega_Q$$

(because $b[0] = 0$). The resulting pure traction problem may be not solvable if certain compatibility conditions are not satisfied (cf. [10, Prop. 4.2]).

1.2 The main result

Our aim is to study the asymptotic behavior of the solution $u[k]$ of problem (1.5) as $k > 0$ approaches zero. More precisely, we plan to apply the *Functional Analytic Approach* of [7] and the periodic elastic single layer potential

$v_q^-[\omega,\cdot]$ (see Section 2) to represent the solution in terms of convergent power series in suitable Banach spaces. To succeed, however, we need an additional assumption on the function $k \mapsto b[k]$. By conditions (1.2) and (1.4) and by the real analyticity of the function $k \mapsto b[k]$ we see that there exists $l \in \mathbb{N} \setminus \{0\}$ such that

- $k \mapsto k^{-l}b[k]$ is real analytic from $]-k_0,k_0[$ to $C^{m-1,\alpha}(\partial\Omega_Q, M_n(\mathbb{R}))$,
- the matrix function $\tilde{b} := \lim_{k\to 0} k^{-l}b[k]$ belongs to $C^{m-1,\alpha}(\partial\Omega_Q, M_n(\mathbb{R}))$ and is not 0,
- $\xi^t\tilde{b}(x)\xi \leq 0$ for all $\xi \in \mathbb{R}^n$ and all $x \in \partial\Omega_Q$.

We shall further assume that

$$\det \int_{\partial\Omega_Q} \tilde{b}\, d\sigma \neq 0. \tag{1.6}$$

Then, with conditions (1.2)–(1.4) and (1.6) we have the following Theorem 1, whose proof we present in the forthcoming sections.

Theorem 1. *There exist a sequence $\{(\hat{\mu}_j, \hat{c}_j)\}_{j\in\mathbb{N}}$ in $C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)_0 \times \mathbb{R}^n$ and a real number $k_\# \in]0,k_0[$ such that*

$$u[k](x) = \sum_{j=0}^{+\infty} v_q^-[\omega,\hat{\mu}_j](x)k^j + \frac{1}{k^l}\sum_{j=0}^{+\infty}\hat{c}_jk^j + Bq^{-1}x \qquad \forall x \in \overline{\mathbb{S}[\Omega_Q]^-}, \quad \forall k \in]0,k_\#[, \tag{1.7}$$

where for all $k \in]-k_\#,k_\#[$ the series $\sum_{j=0}^{+\infty} v_q^-[\omega,\hat{\mu}_j](x)k^j$ converges normally in $C_q^{m,\alpha}(\overline{\mathbb{S}[\Omega_Q]^-},\mathbb{R}^n)$ and $\sum_{j=0}^{+\infty}\hat{c}_jk^j$ converges normally in $\mathbb{R}^n$.

Equation (1.7) shows that $u[k]$ (which is defined only for $k \in]0,k_0[$) can be represented in terms of power series which converge in a whole neighborhood of the degenerate value $k = 0$. From (1.7), we see, for example, that

$$k^lu[k](x) - \hat{c}_0 = k\Big(\sum_{j=0}^{+\infty} v_q^-[\omega,\hat{\mu}_j](x)k^{j+l-1} + \sum_{j=1}^{+\infty}\hat{c}_jk^{j-1} + k^{l-1}Bq^{-1}x\Big) \qquad \forall x \in \overline{\mathbb{S}[\Omega_Q]^-}, \forall k \in]0,k_\#[,$$

and thus that

$$\|k^lu[k] - \hat{c}_0\|_\infty = O(k) \qquad \text{as } k \to 0.$$

We also note that the term $Bq^{-1}x$ in (1.7) shows the effect of the external loading.

In Corollary 2 we will show how to compute the sequence $\{(\hat{\mu}_j, \hat{c}_j)\}_{j\in\mathbb{N}}$ solving certain boundary integral equations and we will see that the term $Bq^{-1}x$ appears also in those equations.

Finally, we observe that the Functional Analytic Approach of this paper has already been used for the analysis of perturbation problems for the Lamé equations in [5,6] for bounded domains and in [10,12] for periodic domains.

2 Preliminaries on periodic potential theory for the Lamé equations

In order to construct the solution of problem (1.5), we will exploit a periodic version of potential theory for the Lamé equations. We say that a function f on $\overline{\mathbb{S}[\Omega_Q]^-}$ is q-periodic if $f(x+qz)=f(x)$ for all $x \in \overline{\mathbb{S}[\Omega_Q]^-}$ and $z \in \mathbb{Z}^n$. To construct periodic elastic layer potentials, we introduce a periodic analog of the fundamental solution of $L[\omega]$ (cf., e.g., Ammari and Kang [1, Lemma 9.21], [10, Thm. 3.1]). So, let

$$\Gamma^q_{n,\omega} := (\Gamma^{q,k}_{n,\omega,j})_{(j,k)\in\{1,\dots,n\}^2}$$

be the $n \times n$ matrix of q-periodic distributions with (j,k) entry defined by

$$\Gamma^{q,k}_{n,\omega,j} := \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{4\pi^2|Q||q^{-1}z|^2}\left[-\delta_{j,k}+\frac{\omega}{\omega+1}\frac{(q^{-1}z)_j(q^{-1}z)_k}{|q^{-1}z|^2}\right]E_{2\pi i q^{-1}z} \qquad \forall (j,k)\in\{1,\dots,n\}^2\,,$$

where $E_{2\pi i q^{-1}z}(x) := e^{2\pi i (q^{-1}z)\cdot x}$ for all $x \in \mathbb{R}^n$ and $z \in \mathbb{Z}^n$. Then,

$$L[\omega]\Gamma^q_{n,\omega} = \sum_{z\in\mathbb{Z}^n} \delta_{qz} I_n - \frac{1}{|Q|} I_n$$

in the sense of distributions, where δ_{qz} denotes the Dirac measure with mass at qz. We mention that similar constructions have been used to define a periodic analog of the fundamental solution for an elliptic differential operator in [7, Chapter 12] and for the heat equation in Luzzini [23]. We set

$$\Gamma^{q,j}_{n,\omega} := \left(\Gamma^{q,j}_{n,\omega,i}\right)_{i\in\{1,\dots,n\}}\,,$$

which we think as column vectors for all $j \in \{1,\dots,n\}$. We now introduce the periodic single layer potential. So, if $\mu \in C^{0,\alpha}(\partial\Omega_Q,\mathbb{R}^n)$, then we denote by $v_q[\omega,\mu]$ the periodic single layer potential, defined as

$$v_q[\omega,\mu](x) := \int_{\partial\Omega_Q} \Gamma^q_{n,\omega}(x-y)\mu(y)\,d\sigma_y \qquad \forall x\in\mathbb{R}^n\,.$$

If $\mu \in C^{0,\alpha}(\partial\Omega_Q,\mathbb{R}^n)$, then $v_q[\omega,\mu]$ is q-periodic and

$$L[\omega]v_q[\omega,\mu] = -\frac{1}{|Q|}\int_{\partial\Omega_Q}\mu\,d\sigma \qquad \text{in } \mathbb{R}^n\setminus\partial\mathbb{S}[\Omega_Q]^-.$$

We set

$$\begin{aligned} V_q[\omega,\mu](x) &:= v_q[\omega,\mu](x) \quad \forall x\in\partial\Omega_Q\,,\\ W^*_q[\omega,\mu](x) &:= \int_{\partial\Omega_Q}\sum_{l=1}^n \mu_l(y)T(\omega, D\Gamma^{q,l}_{n,\omega}(x-y))\nu_{\Omega_Q}(x)\,d\sigma_y \quad \forall x\in\partial\Omega_Q\,. \end{aligned}$$

If $\mu \in C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)$, then $v_q^-[\omega,\mu] := v_q[\omega,\mu]_{|\overline{\mathbb{S}[\Omega_Q]^-}}$ belongs to the Schauder space of q-periodic functions $C_q^{m,\alpha}(\overline{\mathbb{S}[\Omega_Q]^-},\mathbb{R}^n)$ (equipped with its usual norm) and the operator

$$\mu \mapsto v_q^-[\omega,\mu]$$

is continuous from $C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)$ to the space $C_q^{m,\alpha}(\overline{\mathbb{S}[\Omega_Q]^-},\mathbb{R}^n)$. Moreover, the operator

$$\mu \mapsto W_q^*[\omega,\mu]$$

is continuous from the space $C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)$ to itself, and we have

$$T\big(\omega, Dv_q^-[\omega,\mu](x)\big)\nu_{\Omega_Q}(x) = \frac{1}{2}\mu(x) + W_q^*[\omega,\mu](x) \qquad \forall x \in \partial\Omega_Q$$

for all $\mu \in C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)$.

3 Integral equation formulation of (1.5) and proof of Theorem 1

First of all, by exploiting the periodic elastic single layer potential and [8, Thm. 4.4] on the representation of the solution of a Robin-type traction problem in a periodic domain, we immediately deduce the validity of the following proposition where we convert problem (1.5) into an integral equation.

Proposition 1. *Let $k \in]0,k_0[$. Let*

$$C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)_0 := \left\{ f \in C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n) \colon \int_{\partial\Omega_Q} f\,d\sigma = 0 \right\}.$$

Then,

$$u[k](x) = v_q^-[\omega,\mu_k](x) + \frac{c_k}{k^l} + Bq^{-1}x \qquad \forall x \in \overline{\mathbb{S}[\Omega_Q]^-},$$

where (μ_k, c_k) is the unique solution in $C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)_0 \times \mathbb{R}^n$ of

$$\begin{aligned}\frac{1}{2}\mu(x) + W_q^*[\omega,\mu](x) + b[k](x)\Big(V_q[\omega,\mu](x) + \frac{c}{k^l}\Big)& \\ = g(x) - T(\omega,Bq^{-1})\nu_{\Omega_Q}(x) - b[k](x)Bq^{-1}x& \qquad \forall x \in \partial\Omega_Q\,.\end{aligned} \tag{3.1}$$

We introduce the operator Λ from $]-k_0,k_0[$ to the space

$$\mathcal{L}(C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)_0 \times \mathbb{R}^n, C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n))$$

defined by

$$\Lambda[k](\mu,c)(x) := \frac{1}{2}\mu(x) + W_q^*[\omega,\mu](x) + b[k](x)V_q[\omega,\mu](x) + k^{-l}b[k](x)c \qquad \forall x \in \partial\Omega_Q\,,$$

for all $(\mu,c) \in C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)_0 \times \mathbb{R}^n$ and for all $k \in]-k_0,k_0[$. We observe that (3.1) can be rewritten as

$$\Lambda[k](\mu,c)(x) = g(x) - T(\omega,Bq^{-1})\nu_{\Omega_Q}(x) - b[k](x)Bq^{-1}x \qquad \forall x \in \partial\Omega_Q\,.$$

Moreover, for $k = 0$ the linear operator $\Lambda[0]$ becomes

$$\Lambda[0](\mu,c)(x) = \frac{1}{2}\mu(x) + W_q^*[\omega,\mu](x) + \tilde{b}(x)c \qquad \forall x \in \partial\Omega_Q\,, \quad \forall (\mu,c) \in C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)_0 \times \mathbb{R}^n\,, \tag{3.2}$$

and $\Lambda[0]$ is invertible with bounded inverse in the space

$$\mathcal{L}(C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n), C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)_0 \times \mathbb{R}^n)$$

(see [8, Lem. 4.2]). Further properties of $\Lambda[k]$ are presented in the following.

Proposition 2. *The following statements hold.*

(i) The map from $]-k_0,k_0[$ *to* $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)_0 \times \mathbb{R}^n, C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n))$ *that takes* k *to* $\Lambda[k]$ *is real analytic.*

(ii) There exists $k_1 \in]0,k_0[$ *such that for each* $k \in]-k_1,k_1[$ *the linear operator* $\Lambda[k]$ *is invertible with inverse in the space*

$$\mathcal{L}(C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n), C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)_0 \times \mathbb{R}^n)$$

and such that the map from $]-k_1,k_1[$ *to the space*

$$\mathcal{L}(C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n), C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)_0 \times \mathbb{R}^n)$$

that takes k *to* $(\Lambda[k])^{(-1)}$ *is real analytic.*

Proof. The validity of (i) follows by the boundedness of the linear operators $W_q^*[\omega,\cdot]$ and $V_q[\omega,\cdot]$ and by the real analyticity of $k \mapsto k^{-l}b[k]$. To prove (ii), we note that since the set of linear homeomorphisms is open in the set of linear and continuous operators, and since the map that takes a linear invertible operator to its inverse is real analytic (cf. *e.g.*, Hille and Phillips [18, Thms. 4.3.2 and 4.3.4]), there exists $k_1 \in]0,k_0[$ such that the map that takes k to $\Lambda[k]^{(-1)}$ is real analytic from $]-k_1,k_1[$ to $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n), C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)_0 \times \mathbb{R}^n)$. □

By Proposition 2, we represent the solutions of the integral equation (3.1) by means of real analytic maps.

Corollary 1. Let $(\hat{\mu},\hat{c})$ be the real analytic map from $]-k_1,k_1[$ to the space $C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)_0 \times \mathbb{R}^n$ defined by

$$(\hat{\mu}[k],\hat{c}[k]) := (\Lambda[k])^{(-1)}\mathfrak{D}[k]$$

for all $k \in]-k_1,k_1[$, where

$$\mathfrak{D}[k](x) := g(x) - T(\omega, Bq^{-1})\nu_{\Omega_Q}(x) - b[k](x)Bq^{-1}x \quad \forall k \in]-k_0,k_0[\,,\ x \in \partial\Omega_Q\,.$$

Then,

$$(\hat{\mu}[k],\hat{c}[k]) = (\mu_k, c_k)$$

for all $k \in]0,k_1[$ and $(\hat{\mu}[0],\hat{c}[0])$ is the unique solution (μ,c) in the space $C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)_0 \times \mathbb{R}^n$ of

$$\frac{1}{2}\mu(x) + W_q^*[\omega,\mu](x) + \tilde{b}(x)c = \mathfrak{D}[0](x) \qquad \forall x \in \partial\Omega_Q\,. \tag{3.3}$$

Proof. By Proposition 2, $k \mapsto (\hat{\mu}[k], \hat{c}[k])$ is real analytic from $]-k_1, k_1[$ to $C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)_0 \times \mathbb{R}^n$ and $(\hat{\mu}[k], \hat{c}[k]) = (\mu_k, c_k)$ for all $k \in]0, k_1[$. Since $(\hat{\mu}[0], \hat{c}[0]) := (\Lambda[0])^{(-1)}\mathfrak{D}[0]$, by equation (3.2), we deduce that $(\hat{\mu}[0], \hat{c}[0])$ is the unique solution (μ, c) in $C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)_0 \times \mathbb{R}^n$ of equation (3.3). $\square$

Since $k \mapsto b[k]/k^l$ is real analytic, there exist $\tilde{k} \in]-k_0, k_0[$ and a family $\{b_j^\#\}_{j\in\mathbb{N}}$ in $C^{m-1,\alpha}(\partial\Omega_Q, M_n(\mathbb{R}))$ such that $b[k] = k^l \sum_{j=0}^{+\infty} b_j^\# k^j$ for all $k \in]-\tilde{k}, \tilde{k}[$, where the series $\sum_{j=0}^{+\infty} b_j^\# k^j$ converges normally in $C^{m-1,\alpha}(\partial\Omega_Q, M_n(\mathbb{R}))$ for all $k \in]-\tilde{k}, \tilde{k}[$. Possibly taking a smaller $\tilde{k}$, we note that

$$\Lambda[k](\mu, c) = \Lambda[0](\mu, c) + \sum_{j=1}^{+\infty} \left(b_{j-l}^\# V_q[\omega, \mu](x) + b_j^\# c\right) k^j \,,$$

where we understand that $b_{j-l}^\# = 0$ if $j < l$ and where the series

$$\sum_{j=1}^{+\infty} \left(b_{j-l}^\# V_q[\omega, \mu](x) + b_j^\# c\right) k^j$$

converges normally in $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)_0 \times \mathbb{R}^n, C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n))$ for all $k \in]-\tilde{k}, \tilde{k}[$. We find convenient to set

$$\begin{aligned} R_j(\mu, c) &:= b_{j-l}^\# V_q[\omega, \mu](x) + b_j^\# c \qquad \forall j \in \mathbb{N} \setminus \{0\}\,, \\ R[k](\mu, c) &:= \sum_{j=1}^{+\infty} R_j(\mu, c) k^j \,, \end{aligned}$$

and accordingly $\Lambda[k] = \Lambda[0] + R[k]$. By the Neumann series theorem, possibly taking again a smaller $\tilde{k}$, we have

$$(\Lambda[k])^{(-1)} = (\Lambda[0])^{(-1)} + \sum_{r=1}^{+\infty} (-1)^r \left((\Lambda[0])^{(-1)} R[k]\right)^r (\Lambda[0])^{(-1)} \,,$$

where for all $k \in]-\tilde{k}, \tilde{k}[$ the series converges normally in

$$\mathcal{L}(C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n), C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)_0 \times \mathbb{R}^n)\,.$$

For all $r \in \mathbb{N} \setminus \{0\}$, we have

$$\begin{aligned} &\left((\Lambda[0])^{(-1)} R[k]\right)^r \\ &= \sum_{j=1}^{+\infty} \Bigg(\sum_{\substack{j_{l_1}, \dots, j_{l_r} \in \mathbb{N}\setminus\{0\} \\ j_{l_1} + \cdots j_{l_r} = j}} \left((\Lambda[0])^{(-1)} R_{j_1}\right) \cdot \dots \cdot \left((\Lambda[0])^{(-1)} R_{j_r}\right) \Bigg) k^j \,, \end{aligned}$$

where for all $k \in]-\tilde{k}, \tilde{k}[$ the series converges normally in the space

$$\mathcal{L}(C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n), C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)_0 \times \mathbb{R}^n)\,.$$

Then, we set $L_0 := (\Lambda[0])^{(-1)}$ and for each $j \in \mathbb{N} \setminus \{0\}$ we define

$$L_j \in \mathcal{L}(C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n), C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)_0 \times \mathbb{R}^n)$$

as

$$L_j := \sum_{r=1}^{+\infty} (-1)^r \Big(\sum_{\substack{j_{l_1}, \dots, j_{l_r} \in \mathbb{N}\setminus\{0\} \\ j_{l_1}+\dots+j_{l_r}=j}} \Big((\Lambda[0])^{(-1)} R_{j_1} \Big) \cdots \Big((\Lambda[0])^{(-1)} R_{j_r} \Big) \Big) (\Lambda[0])^{(-1)}. \tag{3.4}$$

Accordingly, possibly taking a smaller $\tilde{k}$, one can verify that $(\Lambda[k])^{(-1)} = (\Lambda[0])^{(-1)} + \sum_{j=1}^{+\infty} L_j k^j$, where for all $k \in]-\tilde{k}, \tilde{k}[$ the series converges normally in $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n), C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)_0 \times \mathbb{R}^n)$. Then, we introduce the sequence $\{d_j\}_{j\in\mathbb{N}}$ in $C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)$ by setting

$$\begin{aligned} d_0(x) &:= g(x) - T(\omega, Bq^{-1})\nu_{\Omega_Q}(x) \quad \forall x \in \partial\Omega_Q\,, \\ d_j(x) &:= -\, b^{\#}_{j-l}(x) B q^{-1} x \quad \forall x \in \partial\Omega_Q\,, \forall j \in \mathbb{N}\setminus\{0\}\,, \end{aligned}$$

where we understand that $b^{\#}_{j-l} = 0$ if $j < l$. Possibly shrinking $\tilde{k}$, we note that $\mathfrak{D}[k] = \sum_{j=0}^{+\infty} d_j k^j$, where for all $k \in]-\tilde{k}, \tilde{k}[$ the series converges normally in $C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)$. Then, by the real analyticity of $k \mapsto (\hat{\mu}[k], \hat{c}[k])$ and the expressions for $(\Lambda[k])^{(-1)}$ and for $\mathfrak{D}[k]$, we deduce the following.

Corollary 2. Let

$$(\hat{\mu}_0, \hat{c}_0) = (\Lambda[0])^{(-1)}(d_0), \;\; (\hat{\mu}_j, \hat{c}_j) = \sum_{\substack{j_1, j_2 \in \mathbb{N} \\ j_1+j_2=j}} L_{j_1}(d_{j_2}) \qquad \forall j \in \mathbb{N}\setminus\{0\}\,,$$

where $L_0 := (\Lambda[0])^{(-1)}$ and L_j is as in (3.4) for $j \in \mathbb{N}\setminus\{0\}$. Then, there exists $k_2 \in]0, k_1[$ such that $(\hat{\mu}[k], \hat{c}[k]) = \sum_{j=0}^{+\infty} (\hat{\mu}_j, \hat{c}_j) k^j$ for all $k \in]-k_2, k_2[$, where the series converges normally in $C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)_0 \times \mathbb{R}^n$ for all $k \in]-k_2, k_2[$.

We are now able to prove Theorem 1.
Proof of Theorem 1. We already know that for all $k \in]-k_2, k_2[$ the series $\sum_{j=0}^{+\infty} \hat{c}_j k^j$ converges normally in $\mathbb{R}^n$ and that the series $\sum_{j=0}^{+\infty} \hat{\mu}_j k^j$ converges normally in $C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)_0$. Since $v_q^-[\omega, \cdot]$ is a bounded linear operator from $C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)_0$ to $C_q^{m,\alpha}(\overline{\mathbb{S}[\Omega_Q]^-}, \mathbb{R}^n)$, we deduce that taking a sufficiently small $k_\# \in]0, k_2[$ for all $k \in]-k_\#, k_\#[$ the series $\sum_{j=0}^{+\infty} v_q^-[\omega, \hat{\mu}_j](x) k^j$ converges normally in $C_q^{m,\alpha}(\overline{\mathbb{S}[\Omega_Q]^-}, \mathbb{R}^n)$. Then, the representation formula of Proposition 1 completes the proof. □

Remark 1. If our focus is on the the leading terms of the series that appear in (1.7) we can write

$$u[k](x) = v_q^-[\omega, \hat{\mu}_0](x) + \frac{\hat{c}_0}{k^l} + \sum_{j=1}^{+\infty} v_q^-[\omega, \hat{\mu}_j](x) k^j + \frac{1}{k^l} \sum_{j=1}^{+\infty} \hat{c}_j k^j + Bq^{-1}x$$
$$\forall x \in \overline{\mathbb{S}[\Omega_Q]^-}\,, \forall k \in]0, k_\#[\,,$$

and note that $(\hat{\mu}_0, \hat{c}_0)$ is the unique solution in $C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)_0 \times \mathbb{R}^n$ of the equation

$$\frac{1}{2}\hat{\mu}_0(x) + W_q^*[\omega, \hat{\mu}_0](x) + \tilde{b}(x)\hat{c}_0 = g(x) - T(\omega, Bq^{-1})\nu_{\Omega_Q}(x) \quad \forall x \in \partial\Omega_Q\,.$$

In particular, by arguing as in the proof of [8, Lem. 4.2] and by the identity $\int_{\partial\Omega_Q}(\nu_{\Omega_Q})_j d\sigma{=}0\ \forall j{\in}\{1,\ldots,n\}$, one verifies that $\hat{c}_0{=}\big(\int_{\partial\Omega_Q}\tilde{b}\,d\sigma\big)^{-1}\int_{\partial\Omega_Q} g d\sigma$ and that $\hat{\mu}_0$ is the unique solution in $C^{m-1,\alpha}(\partial\Omega_Q, \mathbb{R}^n)_0$ of

$$\begin{aligned}\frac{1}{2}\hat{\mu}_0(x) + W_q^*[\omega, \hat{\mu}_0](x) =& g(x) - T(\omega, Bq^{-1})\nu_{\Omega_Q}(x)\\ &- \tilde{b}(x)\Big(\int_{\partial\Omega_Q}\tilde{b}\,d\sigma\Big)^{-1}\int_{\partial\Omega_Q} g\,d\sigma \qquad \forall x \in \partial\Omega_Q\,.\end{aligned}$$

4 Conclusions

We have used the Functional Analytic Approach to study the Lamé equations in a periodic domain with a Robin-type boundary condition that turns into a pure traction one. The change in the boundary condition is obtained multiplying the Dirichlet-type term by a k-dependent matrix function $b[k](\cdot)$ that vanishes for $k = 0$. We have seen that for $k > 0$ close to 0 the solution can be written as the sum of two converging power series of k, one being multiplied by the singular function $1/k^l$, and a linear function that takes care of the quasi-periodicity of the solution (and disappears for periodic solutions). The positive natural number l depends on the vanishing order of the matrix $b[k]$ as k tends to 0.

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