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Spectral Method for One Dimensional Benjamin-Bona-Mahony-Burgers Equation Using the Transformed Generalized Jacobi Polynomial

Yu Zhou and Yujian Jiao

Department of Mathematics, Shanghai Normal University
200234 Shanghai, China
E-mail(*corresp.*): yj-jiao@shnu.edu.cn
E-mail: 2643351845@qq.com

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Abstract. The Benjamin-Bona-Mahony-Burgers equation (BBMBE) plays a fundemental role in many application scenarios. In this paper, we study a spectral method for the BBMBE with homogeneous boundary conditions. We propose a spectral scheme using the transformed generalized Jacobi polynomial in combination of the explicit fourth-order Runge-Kutta method in time. The boundedness, the generalized stability and the convergence of the proposed scheme are proved. The extensive numerical examples show the efficiency of the new proposed scheme and coincide well with the theoretical analysis. The advantages of our new approach are as follows: (i) the use of the transformed generalized Jacobi polynomial simplifies the theoretical analysis and brings a sparse discrete system; (ii) the numerical solution is spectral accuracy in space.

Keywords: spectral method, Benjamin-Bona-Mahony-Burgers equation, generalized Jacobi function.

AMS Subject Classification: 76M22; 34G20; 33C45.

1 Introduction

The BBMBE is a mathematical model of propagation of small-amplitude long waves in nonlinear dispersive media. The one-dimensional BBMBE is of the form (cf. [21])

$$\begin{cases} W_t - W_{xxt} - W_{xx} + W_x + WW_x = 0, & x \in (0,L), \quad t \in (0,T], \\ W(0,t) = W(L,t) = 0, & 0 \leqslant t \leqslant T, \\ W(x,0) = W_0(x), & 0 \leqslant x \leqslant L, \end{cases} \tag{1.1}$$

where $W(x,t)$ is the fluid velocity in horizontal direction x, $W_{xxt}(x,t)$ is the dispersive term, $W_{xx}(x,t)$ is the dissipative term, and $W_0(x)$ is the initial state. The dispersive effect of (1.1) is the same as the Benjamin-Bona-Mahony equation [8], while the dissipative effect is the same as the Burgers equation [9]. The BBMBE plays an important role in science and engineering, such as acoustic waves in a harmonic crystal, acoustic–gravity waves in fluids, hydromagnetic waves in cold plasma [1], fissured rock [7], movement of moisture in soil [26]and undular bore [24], and so on. The BBMBE has attracted the interest of many researchers. They have carried out analytically or numerically a great deal of research.

Some authors studied analytical methods for solving the BBMBE. El-Wakil *et al.* [11] used the Exp-function method to obtain the generalized solitary solutions and the periodic solutions for the BBMBE. Ganji *et al.* [13] reported the Exp-Function method for solving the generalized nonlinear BBMBE. Gómez *et al.* [25] obtained traveling wave solutions by a generalization of the tanh–coth method. Yin *et al.* [28] investigated the exponential time decay rate of solutions toward traveling waves for the Cauchy problem of generalized BBMBE by the space-time weighted energy method. Abdollahzadeh *et al.* [2] constructed traveling wave solutions for a special form of the generalized BBMBE using G/G' expansion method. Xiao *et al.* [27] concerned with the convergence rates of the global solutions of the generalized BBMBE to the corresponding degenerate boundary layer solutions in the half-space. Karakoc *et al.* [19] got exact traveling wave solutions by the modified Kudryyashov method and numerical solution by a spetic B-spline collocation finite element method of the BBMBE.

There are many researchers investigated numerical methods for solving the nonlinear BBMBE. Omrani *et al.* [23] proposed a Crank-Nicolson-type finite difference method for solving the one-dimensional BBMBE. Zhang *et al.* [30] proposed the two-level and the three-level linearized finite difference schemes for the BBMBE. Karakoc *et al.* [20] constructed a lumped Galerkin finite element method by cubic B-spline basis for the BBMBE. Ewing [12] studied the Galerkin finite element method for the BBMBE. Arnold *et al.* [4] presented a finite element method for BBMBE, and established optimal order error estimates. Kadri *et al.* [18] studied a fully discrete Galerkin scheme by combining the finite element method and the backward Euler method for the BBMBE. Dehgan *et al.* [10] proposed a meshless method by combining the finite difference method in time direction with the interpolating element-free Galerkin method in spatial direction for solving the nonlinear generalized BBMBE. Al-Khaled *et al.* [3] constructed a decomposition method for the numerical solution of the BBMBE. Zarebnia *et al.* [29] used the cubic B-spline collocation methods to solve the BBMBE. Aslefallah *et al.* [6] studied the singular boundary method for the simulation of nonlinear generalized BBMB problem with initial and

Dirichlet-type boundary conditions. Arora *et al.* [5] developed a hybrid numerical technique combining quintic Hermite collocation method with weighted finite difference scheme to solve the BBMBE. Izadi *et al.* [17] developed a numerical scheme by combination of Taylor series and Boubaker polynomials for solving the BBMBE.

In this work, we study a spectral method using the generalized Jacobi polynomials for the BBMBE (1.1). We analyze the boundedness of solution of the proposed scheme, the generalized stability and the convergence, and present some numerical results.The remainder of the paper is organized as follows. In Section 2, we introduce the transformed generalized Jacobi polynomials, some related properties and approximation results. In Section 3, we propose a spectral scheme for the BBMBE (1.1), prove its boundedness. In Section 4, we analyze the generalized stability and the convergence. In Section 5, we give the details of numerical implementation and some numerical results. The final Section is for the concluding remarks.

2 Preliminaries

In this section, we introduce some results of the generalized Jacobi polynomials, which will be used in the forthcoming discussion. Let $\tilde{\Lambda} = \{\ \xi\ | -1 < \xi < 1\}$ and $\chi(\xi)$ be certain weight function. For an integer $\mu \geq 0$, we denote by $(u,v)_{\mu,\chi,\tilde{\Lambda}}$, $|u|_{\mu,\chi,\tilde{\Lambda}}$ and $\|u\|_{\mu,\chi,\tilde{\Lambda}}$ the inner product, semi-norm and norm of space $H^{\mu}_{\chi}(\tilde{\Lambda})$, respectively. In particular, $H^0_{\chi}(\tilde{\Lambda}) = L^2_{\chi}(\tilde{\Lambda})$ has the inner product $(u,v)_{\chi,\tilde{\Lambda}}$ and the norm $\|u\|_{\chi,\tilde{\Lambda}}$. For simplicity, we omit the subscript χ whenenver $\chi(\xi) \equiv 1$. $\|\cdot\|_{L^\infty}$ respresents the max norm.

Let $\alpha, \beta > -1$ and $\omega^{\alpha,\beta}(\xi) = (1-\xi)^{\alpha}(1+\xi)^{\beta}$ be a weight function. The nth-order Jacobi polynomial defined on $\tilde{\Lambda}$ denoted by $P^{\alpha,\beta}_n(\xi)$. The set of $\{P^{\alpha,\beta}_n(\xi)\}$ is mutually orthogonal with respect to the weight $\omega^{\alpha,\beta}(\xi)$, namely

$$\left(P^{\alpha,\beta}_n(\xi), P^{\alpha,\beta}_m(\xi)\right)_{\omega^{\alpha,\beta}(\xi)} = r^{\alpha,\beta}_n \delta_{nm}, \tag{2.1}$$

where δ_{nm} is the Kronecker Delta symbol, and

$$r^{\alpha,\beta}_n = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}.$$

In particular, the Legendre polynomial $L_n(\xi) = P^{0,0}_n(\xi)$. The set $\{\partial_\xi P^{\alpha,\beta}_n(\xi)\}$ is also orthogonal with respect to the weight $\omega^{\alpha+1,\beta+1}(\xi)$, namely

$$\int_{\tilde{\Lambda}} \partial_\xi P^{\alpha,\beta}_n(\xi) \partial_\xi P^{\alpha,\beta}_m(\xi) \omega^{\alpha+1,\beta+1}(\xi) d\xi = \lambda^{\alpha,\beta}_n r^{\alpha,\beta}_n \delta_{nm}, \tag{2.2}$$

where $\lambda^{\alpha,\beta}_n = n(n+\alpha+\beta+1)$.

Let $\Lambda = [0, L]$. The transformation $\xi = \frac{2}{L}x - 1$ convert the domain $\tilde{\Lambda}$ to the domain Λ. Denote

$$\tilde{P}^{\alpha,\beta}_n(x) = P^{\alpha,\beta}_n(\xi) = P^{\alpha,\beta}_n\left(\frac{2}{L}x - 1\right), \quad \tilde{L}_n(x) = L_n(\xi) = L_n\left(\frac{2}{L}x - 1\right).$$

Obviously, the set $\{\tilde{P}_n^{\alpha,\beta}(x)\}$ is mutually orthogonal with respect to the weight $\tilde{\omega}^{\alpha,\beta}(x) = (2/L)^{\alpha+\beta}\,(L-x)^\alpha x^\beta$, namely

$$\left(\tilde{P}_n^{\alpha,\beta}(x), \tilde{P}_m^{\alpha,\beta}(x)\right)_{\tilde{\omega}^{\alpha,\beta}(x)} = \int_\Lambda \tilde{P}_n^{\alpha,\beta}(x)\tilde{P}_m^{\alpha,\beta}(x)\tilde{\omega}^{\alpha,\beta}(x)dx = \frac{L}{2} r_n^{\alpha,\beta}\delta_{nm}, \quad (2.3)$$

where $r_n^{\alpha,\beta}$ be the same as in (2.1). Thanks to (2.2), we check that

$$\left(\partial_x \tilde{P}_n^{\alpha,\beta}(x), \partial_x \tilde{P}_m^{\alpha,\beta}(x)\right)_{\tilde{\omega}^{\alpha+1,\beta+1}(x)} = \frac{2}{L}\lambda_n^{\alpha,\beta} r_n^{\alpha,\beta}\delta_{nm}.$$

We introduce an orthogonal system $\Phi_n(x)$, which is defined as

$$\Phi_n(x) = \left(4/L^2\right)x(L-x)\tilde{P}_{n-2}^{1,1}(x), \quad n \geqslant 2.$$

It is readily to check that

$$(\Phi_n(x), \Phi_m(x))_{\tilde{\omega}^{-1,-1}(x)} = (L/2) r_{n-2}^{1,1}\delta_{nm}.$$

Clearly, $\Phi_n(0) = \Phi_n(L) = 0$. According to (3.1) in [16], we have the following relations:

$$\Phi_n(x) = \frac{2(n-1)}{2n-1}(\tilde{L}_{n-2}(x) - \tilde{L}_n(x)), \quad (2.4)$$
$$\partial_x \Phi_n(x) = -2(n-1)\tilde{L}_{n-1}(x).$$

Let

$$Q_N(\Lambda) = \mathrm{span}\{\Phi_n(x) \mid 2 \leqslant n \leqslant N\},$$

and

$$H_0^1(\Lambda) = \left\{v \mid v \in H^1(\Lambda) \text{ and } v(0) = v(L) = 0\right\}.$$

The orthogonal projection $\pi_N^{1,0} : H_0^1(\Lambda) \to Q_N(\Lambda)$ is defined by

$$\left(\partial_x(\pi_N^{1,0} v - v), \partial_x \phi\right) = 0, \quad \forall \phi \in Q_N(\Lambda).$$

Lemma 1. ([15]) *If $v \in H_0^1(\tilde{\Lambda})$ and $\|\partial_x^r v\|_{L^2_{\omega^{r-1,r-1}}(\tilde{\Lambda})}$ is finite for $N \geqslant 2$, $1 \leqslant r \leqslant N+1$ and $0 \leq \mu \leq r$, then,*

$$\|\partial_x^\mu(\pi_N^{1,0} v - v)\|_{L^2_{\omega^{\mu-1,\mu-1}}(\tilde{\Lambda})} \leq cN^{\mu-r}\|\partial_x^r v\|_{L^2_{\omega^{r-1,r-1}}(\tilde{\Lambda})}.$$

Through simple derivation of the above inequality, we can get that for $\mu = 0, 1$ and $\forall w \in H_0^1(\Lambda)$

$$\|\partial_x^\mu(\pi_N^{1,0} w - w)\|_{L^2_{\tilde{\omega}^{\mu-1,\mu-1}}(\Lambda)} \leqslant CN^{1-r}\|\partial_x^r w\|_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)}. \quad (2.5)$$

Moreover, according to imbedding theorem and space interpolation we have that for any $w \in H_0^1(\Lambda)$,

$$\|w\|^2_{L^4(\Lambda)} \leqslant \|w\|_{L^2(\Lambda)}\|w\|_{H^1(\Lambda)}. \quad (2.6)$$

3 A spectral scheme for the BBMBE

In this section, we study a spectral scheme for the homogeneous problem of (1.1) and prove its boundedness.

A weak form for problem (1.1) is to find $W \in H^1(0,T;H_0^1(\Lambda)\bigcap H^r(\Lambda)) \cap C^1(0,T;C^2(\Lambda))$ such that

$$\begin{cases} (\partial_t W,\phi)+(\partial_{xt}W,\partial_x\phi)+(\partial_x W,\partial_x\phi)+(\partial_x W,\phi) \\ \qquad\qquad +\,(W\partial_x W,\phi)=0, \quad \phi\in H_0^1(\Lambda), \\ W(x,0)=W_0(x), \quad x\in\Lambda. \end{cases} \tag{3.1}$$

The spectral scheme of (3.1) is to find $w_N \in Q_N(\Lambda)$ for all $0<t\leqslant T$, such that

$$\begin{cases} (\partial_t w_N,\phi_N)+(\partial_{xt}w_N,\partial_x\phi_N)+(\partial_x w_N,\partial_x\phi_N)+(\partial_x w_N,\phi_N) \\ \quad +\,(w_N\partial_x w_N,\phi_N)=0, \qquad\qquad \forall\phi_N\in Q_N(\Lambda),\ t\in(0,T], \\ w_N(x,0)=w_{N,0}=\pi_N^{1,0}W_0(x), \qquad x\in\Lambda. \end{cases} \tag{3.2}$$

For any $w_N \in Q_N(\Lambda)$, we check that

$$(\partial_x w_N, w_N)=\frac{1}{2}\int_\Lambda w_N\partial_x w_N dx=\int_\Lambda d\big(w_N^2\big)=0, \tag{3.3}$$

$$(w_N\partial_x w_N, w_N)=\int_\Lambda w_N^2\cdot\partial_x w_N dx=\frac{1}{3}\int_\Lambda d\big(w_N^3\big)=0. \tag{3.4}$$

Next, we check the boundedness of (3.2). Let

$$E(v,t)=\|v(t)\|_{H^1(\Lambda)}^2+\int_0^t \|\partial_x v(\tau)\|_{L^2(\Lambda)}^2 d\tau. \tag{3.5}$$

Choosing $\phi_N = 2w_N$ in (3.2) and using (3.3) and (3.4), we derive that

$$(\partial_t w_N, 2w_N)+(\partial_{xt}w_N, 2\partial_x w_N)+(\partial_x w_N, 2\partial_x w_N)=0.$$

Then,

$$\partial_t\|w_N(t)\|_{H^1(\Lambda)}^2+2\|\partial_x w_N(t)\|_{L^2(\Lambda)}^2=0. \tag{3.6}$$

Integrating the above equality with respect to t, it yields

$$E(w_N,t)=\|w_N(t)\|_{H^1(\Lambda)}^2+\int_0^t \|\partial_x w_N(t)\|_{L^2(\Lambda)}^2 \le \|w_N(0)\|_{H^1(\Lambda)}^2.$$

Similarly, we can check the boundedness of (3.1).

Let τ be the time step, and $t_n = n\tau$. Integrating (3.6) from 0 to t_n with respect t, we get

$$\int_0^{t_n} \partial_t\|w_N(t)\|_{H^1(\Lambda)}^2 dt+2\int_0^{t_n} \|\partial_x w_N(t)\|_{L^2(\Lambda)}^2 dt=0.$$

Then,

$$\|w_N(t_n)\|_{H^1(\Lambda)}^2+2\int_0^{t_n} \|\partial_x w_N(t)\|_{L^2(\Lambda)}^2 dt=\|w_0\|_{H^1(\Lambda)}^2. \tag{3.7}$$

Let $w_N^n = w_N(t_n)$. According to (3.7), we have

$$\|w_N^n\|^2_{L^2(\Lambda)} \leqslant \|w_0\|^2_{H^1(\Lambda)}, \quad |w_N^n|^2_{H^1(\Lambda)} \leqslant \|w_0\|^2_{H^1(\Lambda)}. \tag{3.8}$$

Thanks to (3.8) and imbedding theorem, we derive that for any $t_n \in [0,T]$,

$$\|w_N^n\|_{L^\infty} \leqslant C\|w_N^n\|_{H^1(\Lambda)} \leqslant C\|w_0\|_{H^1(\Lambda)}.$$

These lead to the following result of discrete conservation law.

Theorem 1. *If w_N^n is a solution of the scheme* (3.2)*, then there exist a positive constant C independent of τ such that*

$$\|w_N^n\|_{L^2(\Lambda)} \leqslant C, \quad |w_N^n|_{H^1(\Lambda)} \leqslant C, \quad \|w_N^n\|_{L^\infty} \leqslant C.$$

4 Error analysis

In this section, we analyze the generalized stability and the convergence of scheme (3.2). We assume that $w_{N,0}$ has the error $\tilde{w}_{N,0}$, which induces the error of numerical solution of (3.2) denoted by $\tilde{w}_N$. Then, we get from (3.2) that

$$\begin{cases} (\partial_t \tilde{w}_N, \phi_N) + (\partial_{xt}\tilde{w}_N, \partial_x\phi_N) + (\partial_x\tilde{w}_N, \partial_x\phi_N) + (\partial_x\tilde{w}_N, \phi_N) \\ \qquad + \left(\dfrac{1}{2}\partial_x\big(\tilde{w}_N(\tilde{w}_N + 2w_N)\big), \phi_N\right) = 0, \qquad \phi_N \in Q_N(\Lambda), \\ \tilde{w}_{N,0} = \tilde{w}_N(x,0), \qquad x \in \Lambda. \end{cases}$$

Choosing $\phi_N = 2\tilde{w}_N$ in (3.2), integrating by parts and using (3.3), it yields

$$\partial_t\|\tilde{w}_N(t)\|^2_{H^1(\Lambda)} + 2|\tilde{w}_N(t)|^2_{H^1(\Lambda)} \le 2|(\tilde{w}_N\partial_x\tilde{w}_N, w_N)|. \tag{4.1}$$

According to Hölder inequality and (2.6), we derive that

$$\begin{aligned} &2|(\tilde{w}_N\partial_x\tilde{w}_N, w_N)| = 2\int_\Lambda |\tilde{w}_N\partial_x\tilde{w}_N w_N|dx \\ &\quad \le 2\Big(\int_\Lambda |\partial_x\tilde{w}_N|^2dx\Big)^{\frac{1}{2}}\Big(\int_\Lambda |\tilde{w}_N|^2|w_N|^2dx\Big)^{\frac{1}{2}} \le \frac{1}{\sqrt{L}}\int_\Lambda |\partial_x\tilde{w}_N|^2dx \\ &\quad + 2\sqrt{L}\int_\Lambda |\tilde{w}_N|^2|w_N|^2dx \le |\tilde{w}_N(t)|^2_{H^1(\Lambda)} + c\|w_N(t)\|^2_{H^1(\Lambda)}\|\tilde{w}_N(t)\|^2_{H^1(\Lambda)}. \end{aligned} \tag{4.2}$$

Substituting (4.2) into (4.1), we get

$$\partial_t\|\tilde{w}_N(t)\|^2_{H^1(\Lambda)} + |\tilde{w}_N(t)|^2_{H^1(\Lambda)} \le c\|w_N(t)\|^2_{H^1(\Lambda)}\|\tilde{w}_N(t)\|^2_{H^1(\Lambda)}.$$

Integrating the above inequality respect to t in $[0,t]$, we have the following result of stability.

Theorem 2. *If $w_N(t)$ is the solution of* (3.2) *and $\tilde{w}_N$ be the error of $w_N(t)$ induced by $\tilde{w}_{N,0}$. We have*

$$E(\tilde{w}_N, t) \leqslant Ce^{\int_0^t \|w_N(\xi)\|^2_{H^1(\Lambda)}d\xi}\|\tilde{w}_0\|^2_{H^1(\Lambda)}.$$

Remark 1. Let f be the source term, and $\tilde{w}_{N,0}$ and $\tilde{f}$ be the errors of $w_{N,0}$ and f. They induce the error of w_N denoted by $\tilde{w}_N$. We can derive similarly the following result of generalized stability of the scheme (5.2):

$$E(\tilde{w}_N,t) \le Ce^{\int_0^t \|w_N(\xi)\|^2_{H^1(\Lambda)}d\xi} \times \Big(\|\tilde{w}_0\|^2_{H^1(\Lambda)} + \int_0^t e^{-\int_0^\xi \|w_N(\eta)\|^2_{H^1(\Lambda)}d\eta}\|\tilde{f}(\xi)\|^2_{L^2(\Lambda)}d\xi\Big).$$

We next consider the convergence of the scheme (3.2). Let $W_N = \pi_N^{1,0}W$ and $\tilde{W}_N = W_N - w_N$. We have from (3.1) that

$$\begin{cases} (\partial_t W_N(t),\phi_N) + (\partial_{xt}W_N(t),\partial_x\phi_N) + (\partial_x W_N(t),\partial_x\phi_N) \\ \quad + (\partial_x W_N(t),\phi_N) + (W_N(t)\partial_x W_N(t),\phi_N) = \sum_{j=1}^{5} H_j(\phi,t), \quad \phi_N \in Q_N(\Lambda), \\ W(x,0) = W_0(x), \quad x\in\Lambda, \end{cases} \tag{4.3}$$

where

$$H_1(\phi_N,t)= (\partial_t(W_N(t)-W(t)),\phi_N),\ H_2(\phi_N,t)= (\partial_{xt}(W_N(t)-W(t)),\partial_x\phi_N),$$
$$H_3(\phi_N,t)= (\partial_x(W_N(t)-W(t)),\partial_x\phi_N),\ H_4(\phi_N,t)= (\partial_x(W_N(t)-W(t)),\phi_N),$$
$$H_5(\phi_N,t) = \big(0.5\partial_x(W_N^2(t)-W^2(t)),\phi_N\big).$$

Subtracting (4.3) from (3.2) leads to

$$\begin{cases} \Big(\partial_t\tilde{W}_N(t),\phi_N\Big) + \Big(\partial_{xt}\tilde{W}_N(t),\partial_x\phi_N\Big) + \Big(\partial_x\tilde{W}_N(t),\partial_x\phi_N\Big) \\ +(\partial_x\tilde{W}_N(t),\phi_N)+(\tilde{W}_N(t)\partial_x\tilde{W}_N(t),\phi_N)= \sum_{j=1}^5 H_j(\phi_N,t),\ \phi_N\in Q_N(\Lambda), \\ \tilde{W}_N(x,0) = W_0(x) - \pi_N^{1,0}W_0(x), \qquad x\in\Lambda. \end{cases} \tag{4.4}$$

Taking $\phi_N = 2\tilde{W}_N(t)$ in (4.4), it yields

$$\partial_t\|\tilde{W}_N(t)\|^2_{H^1(\Lambda)} + 2|\tilde{W}_N(t)|^2_{H^1(\Lambda)} \leqslant |(2W_N(t)\tilde{W}_N(t),\partial_x\tilde{W}_N(t))| + \sum_{j=1}^5 |H_j(2\tilde{W}_N(t),t)|. \tag{4.5}$$

By Hölder inequality, (2.6) and (2.5) with $\mu = r = 1$, we have

$$\Big|\Big(2W_N(t)\tilde{W}_N(t),\partial_x\tilde{W}_N(t)\Big)\Big| \le C\Big(\int_\Lambda |\partial_x\tilde{W}_N(t)|^2dx\Big)^{\frac{1}{2}} \times\Big(\int_\Lambda |\tilde{W}_N(t)|^2|W_N(t)|^2dx\Big)^{\frac{1}{2}} \leqslant \frac{1}{6}\int_\Lambda |\partial_x\tilde{W}_N(t)|^2dx + C\|W_N(t)\|^2_{H^1(\Lambda)} \times\|\tilde{W}_N(t)\|^2_{H^1(\Lambda)} \le \frac{1}{6}|\tilde{W}_N(t)|^2_{H^1(\Lambda)} + C\|W(t)\|^2_{H^1(\Lambda)}\|\tilde{W}_N(t)\|^2_{H^1(\Lambda)}. \tag{4.6}$$

Thanks to Hölder inequality and (2.5) with $\mu = 0, 1$, we check that

$$\big|H_1(2\tilde{W}_N(t),t)\big| = \Big|2\int_\Lambda \partial_t(W_N(t)-W(t))\tilde{W}_N(t)dx\Big|$$

$$
\begin{aligned}
&\le C\Big(\int_\Lambda |\tilde{W}_N(t)|^2dx\Big)^{\frac{1}{2}}\Big(\int_\Lambda |\partial_t(W_N(t)-W(t))|^2dx\Big)^{\frac{1}{2}}\\
&\le \frac{1}{6}|\tilde{W}_N(t)|^2_{H^1(\Lambda)}+\|\partial_t(W_N(t)-W(t))\|^2_{L^2(\Lambda)}\\
&\le \frac{1}{6}|\tilde{W}_N(t)|^2_{H^1(\Lambda)}+CN^{-2r}\|\partial_x^r(\partial_tW(t))\|^2_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)},
\end{aligned}
$$

and

$$
\begin{aligned}
|H_2(2\tilde{W}_N(t),t)| &= \Big|2\int_\Lambda \partial_{xt}(W_N(t)-W(t))\partial_x\tilde{W}_N(t)dx\Big|\\
&\le C\Big(\int_\Lambda |\partial_x\tilde{w}_N(t)|^2dx\Big)^{\frac{1}{2}}\Big(\int_\Lambda |\partial_{xt}(W_N(t)-W(t))|^2dx\Big)^{\frac{1}{2}}\\
&\le \frac{1}{6}|\tilde{W}_N(t)|^2_{H^1(\Lambda)}+C\|\partial_{xt}(W_N(t)-W(t))\|^2_{L^2(\Lambda)}\\
&\le \frac{1}{6}|\tilde{W}_N(t)|^2_{H^1(\Lambda)}+CN^{2-2r}\|\partial_x^r(\partial_tW(t))\|^2_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)}.
\end{aligned}
$$

By integrating by parts, Hölder inequality and (2.5) with $\mu=1,0$, we get

$$
\begin{aligned}
|H_3(2\tilde{W}_N(t),t)| &= 2\Big|\int_\Lambda \partial_x(W_N(t)-W(t))\partial_x\tilde{W}_N(t)dx\Big|\\
&\le C\Big(\int_\Lambda |\partial_x\tilde{W}_N(t)|^2dx\Big)^{\frac{1}{2}}\Big(\int_\Lambda |\partial_x(W_N(t)-W(t))|^2dx\Big)^{\frac{1}{2}}\\
&\le \frac{1}{6}|\tilde{W}_N(t)|^2_{H^1(\Lambda)}+C\|\partial_x(W_N(t)-W(t))\|^2_{L^2(\Lambda)}\\
&\le \frac{1}{6}|\tilde{W}_N(t)|^2_{H^1(\Lambda)}+CN^{2-2r}\|\partial_x^rW(t)\|^2_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)},
\end{aligned}
$$

and

$$
\begin{aligned}
|H_4(2\tilde{W}_N(t),t)| &= 2|\int_\Lambda \partial_x(W_N(t)-W(t))\tilde{W}_N(t)dx|\\
&= 2|\int_\Lambda (W_N(t)-W(t))\partial_x\tilde{W}_N(t)dx|\\
&\le C\Big(\int_\Lambda |\partial_x\tilde{W}_N(t)|^2dx\Big)^{\frac{1}{2}}\Big(\int_\Lambda |W_N(t)-W(t)|^2dx\Big)^{\frac{1}{2}}\\
&\le \frac{1}{6}|\tilde{W}_N(t)|^2_{H^1(\Lambda)}+C\|W_N(t)-W(t)\|^2_{L^2(\Lambda)}\\
&\le \frac{1}{6}|\tilde{W}_N(t)|^2_{H^1(\Lambda)}+CN^{-2r}\|\partial_x^rW(t)\|^2_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)}.
\end{aligned}
$$

Using Hölder inequality, (2.6), (2.5) with $\mu=r=1$ and (2.5) with $\mu=1$, we derive that

$$
\begin{aligned}
&|H_5(2\tilde{W}_N(t),t)| = |(W_N^2(t)-W^2(t),\partial_x\tilde{W}_N(t))|\\
&\quad = |(W_N^2(t)-W_N(t)W(t)+W_N(t)W(t)-W^2(t),\partial_x\tilde{W}_N(t))|\\
&\quad \le |(W_N^2(t)-W_N(t)W(t),\partial_x\tilde{W}_N(t))|+|(W_N(t)W(t)-W^2(t),\partial_x\tilde{W}_N(t))|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{6}|\tilde{W}_N(t)|^2_{H^1(\Lambda)} + C\Big(\int_\Lambda |W_N(t)|^2|W_N(t)-W(t)|^2dx \\
&\quad + \int_\Lambda |W(t)|^2|W_N(t)-W(t)|^2dx\Big) \le \frac{1}{6}|\tilde{W}_N(t)|^2_{H^1(\Lambda)} \\
&\quad + C\big(\int_\Lambda |W_N(t)-W(t)|^4dx\big)^{\frac{1}{2}}\Big(\big(\int_\Lambda |W_N(t)|^4dx\big)^{\frac{1}{2}} + \big(\int_\Lambda |W(t)|^4dx\big)^{\frac{1}{2}}\Big) \\
&\quad \le \frac{1}{6}|\tilde{W}_N(t)|^2_{H^1(\Lambda)} + C\|W_N(t)-W(t)\|^2_{H^1(\Lambda)}\|W(t)\|^2_{H^1(\Lambda)} \\
&\quad \le \frac{1}{6}|\tilde{W}_N(t)|_{H^1(\Lambda)} + CN^{2-2r}\|\partial_x^r W(t)\|^2_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)}\|W(t)\|^2_{H^1(\Lambda)}.
\end{aligned}
$$

Substituting (4.6) and the above estimations of $H_j, 1 \le j \le 5$ into (4.5), we get

$$
\begin{aligned}
&\partial_t\|\tilde{W}_N(t)\|^2_{H^1(\Lambda)} + |\tilde{W}_N(t)|^2_{H^1(\Lambda)} \\
&\le C\|W(t)\|^2_{H^1(\Lambda)}\|\tilde{W}_N(t)\|^2_{H^1(\Lambda)} + CN^{2-2r}A_r(W(t)),
\end{aligned} \tag{4.7}
$$

where

$$
\begin{aligned}
A_r(W(t)) =& \|\partial_x^r(\partial_t W(t))\|^2_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)} + \|\partial_x^r W(t)\|^2_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)} \\
&+ \|\partial_x^r W(t)\|^2_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)}\|W(t)\|^2_{H^1(\Lambda)}.
\end{aligned}
$$

Let $E(v,t)$ be the same as defined in (3.5). We get from (4.7) that

$$
\partial_t\big(E(\tilde{W}_N(t),t)e^{-\int_0^t \|W(\xi)\|^2_{H^1(\Lambda)}d\xi}\big) \le CN^{2-2r}e^{-\int_0^t \|W(\xi)\|^2_{H^1(\Lambda)}d\xi}A_r^2(W(t)).
$$

Integrating the above inequality with respect to t in $[0,t]$ and using (2.5), we deduce that

$$
\begin{aligned}
&E(\tilde{W}_N(t),t) \le CN^{2-2r}e^{\int_0^t \|W(\xi)\|^2_{H^1(\Lambda)}d\xi} \\
&\quad \times \Big(\int_0^t e^{-\int_0^\xi \|W(\eta)\|^2_{H^1(\Lambda)}d\eta}A_r^2(W(\xi))d\xi + \|\partial_x^r W(0)\|^2_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)}\Big).
\end{aligned} \tag{4.8}
$$

Thanks to (2.5) , we have

$$
\begin{aligned}
&E(W_N(t)-W(t),t) \\
&\quad \le CN^{2-2r}\Big(\|\partial_x^r W(t)\|^2_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)} + \int_0^t \|\partial_\xi^r W(\xi)\|^2_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)}d\xi\Big).
\end{aligned} \tag{4.9}
$$

A combination of (4.8) and (4.9) leads to the following result of convergence:

Theorem 3. *Let $W(x,t)$ and $w_N(x,t)$ be the solutions of* (3.1) *and* (3.2)*, then for integers $N \ge 2$ and $1 \le r \le N+1$,*

$$
\begin{aligned}
E(w_N - W,t) \le& CN^{2-2r}\Big(e^{\int_0^t \|W(\xi)\|^2_{H^1(\Lambda)}d\xi}\big(\int_0^t e^{-\int_0^\xi \|W(\eta)\|^2_{H^1(\Lambda)}d\eta}A_r^2(W(\xi))d\xi \\
&+ \|\partial_x^r W(0)\|^2_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)}\big) + \|\partial_x^r W(t)\|^2_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)} \\
&+ \int_0^t \|\partial_\xi^r W(\xi)\|^2_{L^2_{\tilde{\omega}^{r-1,r-1}}(\Lambda)}d\xi\Big),
\end{aligned}
$$

provided that the norms appearing at the right side of the above inequality are finite.

5 Numerical results

In this section, we describe the details of the numerical implementation and present some numerical results. The numerical solution of the BBMBE (1.1) can be expressed as follows:

$$w_N(x,t)=\sum_{i=2}^{N}\hat{w}_i(t)\Phi_i(x).$$

Taking $\phi_N=\Phi_j(x)$ in (3.2), we have that

$$\begin{aligned}&\sum_{i=2}^{N}\partial_t\hat{w}_i(t)\left(\Phi_i(x),\Phi_j(x)\right)+\sum_{i=2}^{N}\partial_t\hat{w}_i(t)\left(\partial_x\Phi_i(x),\partial_x\Phi_j(x)\right)\\&+\sum_{i=2}^{N}\hat{w}_i(t)\left(\partial_x\Phi_i(x),\partial_x\Phi_j(x)\right)+\sum_{i=2}^{N}\hat{w}_i(t)\left(\partial_x\Phi_i(x),\Phi_j(x)\right)\\&+\left(w_N(t)\partial_x w_N(t),\Phi_j(x)\right)=0,\quad 2\leqslant j\leqslant N.\end{aligned}\tag{5.1}$$

We introduce vectors $\mathbf{Y}(t)=(\hat{w}_2(t),\dots,\hat{w}_N(t))^T$, $\mathbf{N}(t)=(n_2(t),\dots,n_N(t))^T$ and matrices $\mathbf{M}=(m_{j,i})$, $\mathbf{S}=(s_{j,i})$, $\mathbf{D}=(d_{j,i})$. Thanks to (2.4) and (2.3) with $\alpha=\beta=0$, we deduce that

$$m_{j,i}=(\Phi_i(x),\Phi_j(x))=\begin{cases}\dfrac{4L(j-1)(j-3)}{(2j-5)(2j-3)(2j-1)}, & i=j-2,\\ \dfrac{8L(j-1)^2(2j-1)}{(2j-3)(2j-1)^2(2j+1)}, & i=j,\\ -\dfrac{4L(j-1)(j+1)}{(2j-1)(2j+1)(2j+3)}, & i=j+2,\\ 0, & \text{otherwise}.\end{cases}$$

$$s_{j,i}=(\partial_x\Phi_i(x),\partial_x\Phi_j(x))=\begin{cases}\dfrac{4L(j-1)^2}{(2j-1)}, & i=j,\\ 0, & \text{otherwise}.\end{cases}$$

$$d_{j,i}=(\partial_x\Phi_i(x),\Phi_j(x))=\begin{cases}-\dfrac{4L(j-2)^2}{(2j-3)(2j-1)}, & i=j-1,\\ \dfrac{4L(j-1)j}{(2j-1)(2j+1)}, & i=j+1,\\ 0, & \text{otherwise}.\end{cases}$$

$$n_j(t)\approx\sum_{k=0}^{N}w_N(x_k,t)\partial_x w_N(x_k,t)\Phi_j(x_k)\omega_k,\quad 2\le j\le N,$$

where $x_k=\frac{L}{2}(\xi_k+1)$, $\{\xi_k,\omega_k\}_{k=0}^N$ be the Legendre-Gauss nodes and weights. The system (5.1) can be written in compact form:

$$(\mathbf{M}+\mathbf{S})\partial_t\mathbf{Y}(t)+(\mathbf{S}+\mathbf{D})\mathbf{Y}(t)+\mathbf{N}(t)=0.$$

In order to compute numerical errors, we add a source term $f(x,t)$ to the BBMBE (1.1). Thus, we have the following scheme with the source term:

$$\begin{cases} (\partial_t w_N, \phi_N) + (\partial_{xt} w_N, \partial_x \phi_N) + (\partial_x w_N, \partial_x \phi_N) + (\partial_x w_N, \phi_N) \\ \qquad + (w_N \partial_x w_N, \phi_N) = (f, \phi_N), \qquad \forall \phi_N \in Q_N(\Lambda),\ t \in (0,T], \\ w_{N,0} = w_N(x,0), \qquad x \in \Lambda. \end{cases} \tag{5.2}$$

The corresponding compact form is

$$(\mathbf{M} + \mathbf{S})\partial_t \mathbf{Y}(t) + (\mathbf{S} + \mathbf{D})\mathbf{Y}(t) + \mathbf{N}(t) = \mathbf{F}(t), \tag{5.3}$$

where $\mathbf{F}(t) = (f_2(t), \dots, f_N(t))^T$ with entries

$$f_j(t) \approx \sum_{k=0}^{N} f(x_k,t)\Phi_j(x_k)\omega_k, \quad 2 \le j \le N.$$

We use the explicit fourth-order Runge-Kutta scheme with step τ to discrete the system (5.3) and the discrete $L^\infty-$error norm to describe the numerical error:

$$E_{N,\tau}(t) = \|W - w_N\|_{L^\infty} = \max_{0 \le k \le N} |W(\xi_k,t) - w_N(\xi_k,t)|.$$

Example 1. Taking the test function

$$W(x,t) = \sin(\pi t)\sin(\pi x). \tag{5.4}$$

Table 1. Errors $\log_{10} E_{N,\tau}(t)$ of scheme (5.2) with test function (5.4).

	$\tau = 0.01$	$\tau = 0.001$	$\tau = 0.0001$
N=5	1.46E-4	1.46E-4	1.46E-4
N=10	1.40E-9	1.65E-9	1.65E-9
N=15	5.01E-10	4.86E-14	9.79E-15
N=20	5.04E-10	5.00E-14	1.77E-16
N=25	5.06E-10	5.04E-14	1.75E-16

In Table 1, we tabulate the errors versus the modes N with $T = 1$, $L = 2$ and $\tau = 0.01, 0.001, 0.0001$. The data shows that the errors decay exponentially. In Figure 1, we plot the errors for $0 \le t \le 100$ with $\tau = 0.001$, $N = 20$ and $L = 2$. The profile indicates the stability of long-time computation.

Example 2. We take test function

$$W(x,t) = \exp(-t)\sin(\pi x). \tag{5.5}$$

In Table 2, we list the errors *vs.* modes N with $T = 1$, $L = 2$ and $\tau = 0.01, 0.001, 0.0001$. This indicate that the spectral accuracy is achieved.

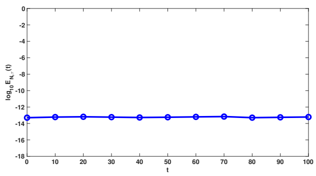


Figure 1. Errors of of scheme (5.2) with test function (5.4).

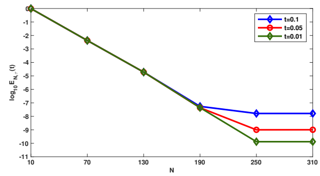
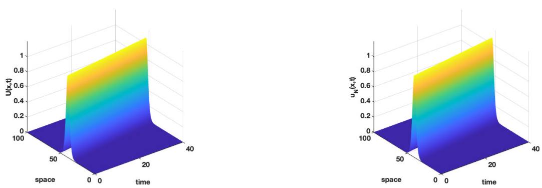


Figure 2. Errors of scheme (5.2) with test function (5.6).

Table 2. Errors $\log_{10} E_{N,\tau}(t)$ of scheme (5.2) with test function (5.5).

	$\tau = 0.01$	$\tau = 0.001$	$\tau = 0.0001$
N=5	2.22E-00	2.22E-00	2.22E-00
N=15	2.36E-01	2.36E-01	2.36E-01
N=25	1.21E-05	1.21E-05	1.21E-05
N=35	1.53E-11	1.52E-11	1.53E-11
N=45	3.57E-11	5.11E-15	4.55E-15



(a) exact solution (b) numerical solution

Figure 3. Time evolution of the exact and the numerical solutions.

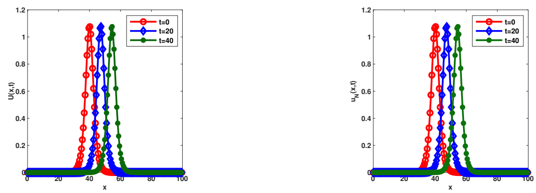
Example 3. We consider the test function

$$W(x,t) = 12\kappa^2 sech^2(\kappa(x - 4\kappa^2 t - x_0)). \tag{5.6}$$

Remark 2. In Example 3, since the test function (5.6) decays exponentially as $|x - x_0| \to \infty$. In actual computation, we choose $L = 100$ and $x_0 = 40$. Thus, (5.6) satisfies the homogeneous boundary condition approximately.

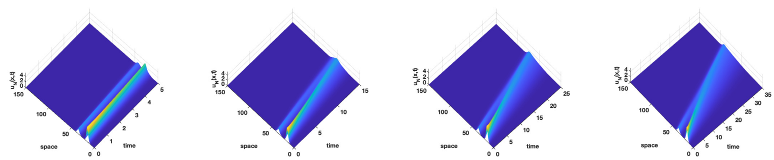
In Figure 2, we sketch the errors vs. the modes N with $T = 1$, $L = 100$, $x_0 = 40$ and $\tau = 0.1, 0.05, 0.01$. The curves show that the numerical solutions converge exponentially. In Figure 3, we depict the time evolution of the exact solution (5.6) (a) and the corresponding numerical solution (b) with $L = 100$, $x_0 = 40$ and $\kappa = 0.3$. In Figure 4, we draw the profiles of the exact solution (a) and the numerical solution (b) at $t = 0, 20, 40$ with $L = 100$, $x_0 = 40$, $\kappa = 0.3$ and $N = 250$. These figures imply that the numerical solution approximate the exact solution well.

Example 4. In this example, we cosider the scheme (3.2) with the initial condi-



(a) exact solution (b) numerical solution

Figure 4. Time evolution of the exact and the numerical solutions.



(a) T=5 (b) T=15 (c) T=25 (d) T=35

Figure 5. Time evolution of the numerical solution of scheme (3.2) with initial condition (5.7).

tion:

$$w(x,0) = \sum_{j=1}^{2} 3d_j sech^2(k_j(x - x_j)), \tag{5.7}$$

where $k_1 = 0.4$, $k_2 = 0.3$, $x_1 = 15$, $x_2 = 35$ and $d_j = \dfrac{4k_j^2}{1 - 4k_j^2}$, $j = 1, 2$.

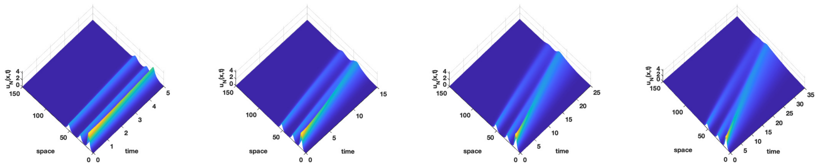
In Figure 5, we draw the time evolution of two solitary waves with $L = 150$, $N = 300$ and $t = 5, 15, 25, 35$. The pictures show that two waves merge together when they encounter. The two solitary waves are not preserved as they do in the BBM equation [22].

Example 5. In this example, we consider the scheme (3.2) with the initial condition:

$$w(x,0) = \sum_{j=1}^{3} 3d_j sech^2(k_j(x - x_j)), \tag{5.8}$$

where $k_1 = 0.39$, $k_2 = 0.3$, $k_3 = 0.3$, $x_1 = 10$, $x_2 = 28$, $x_3 = 52$ and $d_j = \dfrac{4k_j^2}{1 - 4k_j^2}$, $j = 1, 2, 3$.

In Figure 6, we capture the time evolution of three solitary waves with $L = 150$, $N = 300$ and $t = 5, 15, 25, 35$. The snapshots show also that three waves merge together when they encounter.



(a) T=5 (b) T=15 (c) T=25 (d) T=35

Figure 6. Time evolution of the numerical solution of scheme (3.2) with initial condition (5.8).

6 Conclusions

In this paper, we proposed a spectral method for the BBMBE with Dirichlet boundary conditions. In order to fit the problem well we transformed the generalized Jacobi ploynomial to the interval $[0, L]$ and give some related properties. We proposed a spectral method and analyzed its boundedness. Because of the poblem is nonlinear, we proved the generalized stability as described in [14] and the convergence. We gave the details of numerical implementation and some numerical results. This new approach has some advantages: (i) The transformed Jacobi polynomial simulated the BBMBE defined on the interval $[0, L]$ naturally and simplified the theoretical analysis. (ii) The use of the transformed Jacobi polynomial brought a sparse discrete system which can be inversed efficiently. (iii) The spectal accuracy in space was achieved. Furthermore, we will consider one-dimensional BBMBE with other boundary conditions and two/three-dimensional BBMBE.

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