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Numerical Study of the Rosensweig Instability in a Magnetic Fluid Subject to Diffusion of Magnetic Particles

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Abstract. The present study is devoted to the classical problem on stability of a magnetic fluid layer under the influence of gravity and a uniform magnetic field. A periodical peak-shaped stable structure is formed on the fluid surface when the applied magnetic field exceeds a critical value. The mathematical model describes a single peak in the pattern assuming axial symmetry of the peak shape. The field configuration in the whole space, the magnetic particle concentration inside the fluid and the free surface structure are unknown quantities in this model. The unknown free surface is treated explicitly, using a parametric representation with respect to the arc length. The nonlinear problem is discretized by means of a finite element method for the Maxwell's equations and a finite-difference method for the free surface equations. Numerical modelling allows to get over-critical equilibrium free surface shapes in a wide range of applied field intensities. Our numerical results show a significant influence of the particle diffusion on the overcritical shapes.

Keywords: magnetic fluid, particle diffusion, equilibrium free surface, finite element method, finite-difference scheme.

AMS Subject Classification: 65L12; 65N30; 70C20; 76D45; 76E17.

1 Introduction

Magnetic fluids are stable colloidal suspensions of ferromagnetic nano-particles (of size 3-15 nm) in a carrier liquid (water, oil, bio-compatible liquid and others). On a macroscopic level magnetic fluids can be considered as incompressible and nonconducting continuous media. A unique property of magnetic fluids is the combination of fluidity and strong interaction with magnetic fields.

The present study is devoted to the classical problem of ferrohydrostatics on stability (known as the normal field instability or the Rosensweig instability) of a horizontal semi-infinite layer of a magnetic fluid under the influence of gravity and a uniform magnetic field normal to the plane free surface of the layer [13]. A periodical peak-shaped structure is formed on the fluid surface when the applied magnetic field exceeds a critical value. A modified free surface of the layer presents a new static state. This phenomenon was observed first experimentally [5]. Later, the occurrence of regular hexagonal and square pattern of peaks was theoretically predicted in [6], using the energy minimization principle. A quantitative comparison between experiment and numerical simulation of the Rosensweig instability can be found in [7]. Up to now all analytical and numerical investigations of the Rosensweig instability assumed a uniform ferromagnetic particle distribution in the bulk of the magnetic fluid for any applied field intensity.

In order to reach the equilibrium between concentration and the magnetic field, quite a long time (up to days) is needed. The concentration remains almost constant for much shorter time scales. That is why, the validity of the results presented in [7] are not abolished by the present paper. Long-term experiments of the Rosensweig instability for measuring the particle density would be desirable.

A contemporary mathematical modelling of hydrostatics problems for a magnetic fluid should take into account the process of diffusion of ferromagnetic particles under the action of nonuniform magnetic fields. The main goal of the present study is to investigate the influence of the particle diffusion on a free magnetic-fluid surface shape in the case of uniform external magnetic fields. A uniformity of the external field is disturbed by the presence of the magnetic fluid. We perform a mathematical modeling of the Rosensweig instability phenomenon, taking into consideration the particle inhomogeneity in the magnetic fluid.

A spatially varying concentration presents a new unknown quantity of the model. The nonlinear problem is discretized by the finite element method for the Maxwell's equations and the finite-difference method for the free surface equations. The finite element method proves to be a powerful tool for a numerical treatment of free surface problems in ferrohydrodynamics [9].

2 Mathematical Modelling

We consider a semi-infinite magnetic-fluid layer with a horizontal plane free surface bounded from above by a nonmagnetic gas (air). The system is regarded under the action of gravity and a uniform magnetic field normal to the plane free surface of the layer. The magnetic fluid consists of spherical non-interacting particles of a ferromagnetic material. The particles are uniformly distributed inside of the unperturbed fluid layer with the constant volumetric concentration C_0. The initially unperturbed plane free surface is defined by the equation $z = 0$. The applied magnetic field $\boldsymbol{H}_0$ is parallel to the z-axis, i.e., $\boldsymbol{H}_0 = (0, 0, H_0)$. A two-dimensional cut through the perturbed three-dimensional layer surface is shown in Fig. 1. Here a denotes half of the distance between two nearest peaks in the pattern.

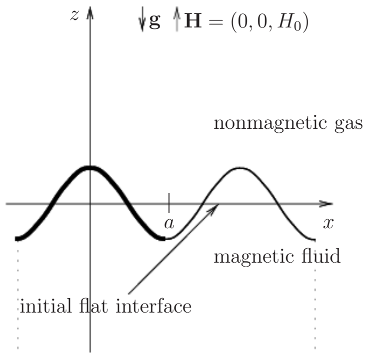

Figure 1. Illustration of the surface deformation.

A simplified mathematical model can be derived for a single peak in the pattern under the assumption of axial symmetry. It allows us to formulate the model in cylindrical coordinates (r, z) with a range of definition $[0, a] \times [-\infty, +\infty]$. This model was successfully applied in [2] for the numerical study of the Rosensweig instability under a uniform concentration approximation.

The applied uniform field $\boldsymbol{H}_0$ is perturbed only in a neighbourhood of the interface Γ between fluid and air. The magnetic field approximates the vertically directed uniform field far from the interface both inside and outside the fluid. The unbounded domain in z-direction can be restricted to a bounded one by introducing the asymptotic boundaries $z = \pm\delta a$. Of course, the distances of these boundaries from the interface should be large enough.

The mathematical model for a diffusion process of ferromagnetic particles in a magnetic fluid with a free surface leads to a coupled problem formulation consisting of three subproblems. The first subproblem describes the magnetic field structure inside the fluid and in the surrounding air by the Maxwell's equations. The second subproblem concerns the diffusion of particles in the bulk of the fluid as a steady-state concentration problem. Finally, the third subproblem is given by the generalized Young-Laplace equation for the axisymmetric equilibrium free-surface shape of the peak. The model equations are formulated in dimensionless variables with characteristic scales H_0, C_0 and a for the magnetic field $\boldsymbol{H}$, the particle concentration C and space variables (r, z), respectively.

The Maxwell's equations inside the magnetic fluid and in the air are

$$\nabla \times \boldsymbol{H} = \boldsymbol{0}, \quad \nabla \cdot \left[\left(1 + \frac{M(H, C)}{H_0 H} \right) \boldsymbol{H} \right] = 0$$

where $M = M(H, C)$ denotes the magnetisation, H the magnetic field intensity. The concentration C is determined by the diffusion of magnetic particles and equals one in uniform magnetic fields. The magnetisation of air is equal to zero. A magnetic-fluid magnetisation follows the Langevin equation. In magnetically dilute fluids, the magnetisation can be assumed to be proportional to the particle concentration, i.e.

$$M(H,C) = M_s C L(\gamma H), \quad L(t) = \coth(t) - \frac{1}{t}, \quad \gamma = \frac{3\chi}{M_s} H_0. \tag{2.1}$$

Here M_s are the saturation magnetisation and χ the initial susceptibility of the fluid. We express the magnetic field in terms of a scalar magnetic potential ϕ as $\boldsymbol{H} = \nabla\phi$. The Maxwell's equations are reformulated in the form

$$\nabla \cdot (\mu(C, |\nabla\phi_1|)\nabla\phi_1) = 0, \quad \mu = 1 + \frac{3\chi}{\gamma}\frac{CL(\gamma|\nabla\phi_1|)}{|\nabla\phi_1|} \quad \text{in the fluid,} \tag{2.2}$$

$$\nabla \cdot (\nabla\phi_2) = 0 \quad \text{in the air.} \tag{2.3}$$

The subscripts 1 and 2 denote variables in the fluid and in the air, respectively. The magnetic potential satisfies the transition boundary conditions

$$\phi_1 = \phi_2, \quad \mu(C, |\nabla\phi_1|)\frac{\partial\phi_1}{\partial n} = \frac{\partial\phi_2}{\partial n} \qquad \text{at } \Gamma \tag{2.4}$$

where Γ is the interface between the magnetic and the non-magnetic media, see [13], and $\boldsymbol{n}$ is a unit normal vector. Symmetry conditions

$$\frac{\partial\phi_1}{\partial r} = 0, \quad \frac{\partial\phi_2}{\partial r} = 0 \quad \text{for } r = 0,\ z \in [-\delta, \delta], \tag{2.5}$$

$$\frac{\partial\phi_1}{\partial r} = 0, \quad \frac{\partial\phi_2}{\partial r} = 0 \quad \text{for } r = 1,\ z \in [-\delta, \delta] \tag{2.6}$$

are set at the peak axis ($r = 0$) and at the peak foot ($r = 1$). Far from the interface Γ the magnetic field takes constant value h_0^1 inside the fluid and equals one in the air. This means

$$\phi_1 = h_0^1 z \quad \text{for } z = -\delta, \quad \phi_2 = z \quad \text{for } z = \delta. \tag{2.7}$$

To define h_0^1, conditions (2.4) are considered at the undisturbed interface $z = 0$

$$\mu(1, |\nabla\phi_1|)\frac{\partial\phi_1}{\partial z} = \frac{\partial\phi_2}{\partial z} \quad \Rightarrow \quad \mu(1, h_0^1)h_0^1 = 1 \quad \Rightarrow \quad h_0^1 + \frac{3\chi}{\gamma}L(\gamma h_0^1) = 1.$$

The magnetic field structure is defined in terms of the magnetic potential by equations (2.2),(2.3) with boundary conditions (2.4)–(2.7) formulated in the bounded rectangular domain $[0, 1] \times [-\delta, \delta]$.

The second subproblem of the mathematical model describes a magnetophoresis process, i.e. the diffusion of ferromagnetic particles in the magnetic fluid under the action of a nonuniform magnetic field. This process was mathematically studied in [11]. An explicit analytical solution for the steady-state particle concentration problem was constructed

$$C = C(\gamma H) = \frac{\psi(\gamma H)V}{\int\limits_{\Omega_f} \psi(\gamma H)d\Omega_f}, \quad \psi(t) = \exp\Big(\int_0^t L(x)dx\Big) = \frac{\sinh(t)}{t} \quad \text{in } \Omega_f.$$

Here V is the volume of the fluid and Ω_f denotes a domain, filled with the fluid. As a result, we can write an exact solution for the particle concentration problem in the magnetic-fluid layer

$$C(\gamma H)=\frac{\delta}{I}\psi(\gamma H),\ I=\frac{1}{\pi}\int\limits_{\Omega_f}\psi(\gamma H)d\Omega_f=2\int\limits_{\Omega_f}\frac{\sinh(\gamma H)}{\gamma H}rdrdz \text{ in } \Omega_f. \quad (2.8)$$

Equilibrium shapes of a free magnetic-fluid surface are described by the generalized Young-Laplace equation

$$\sigma\frac{\mathcal{K}}{a}=-a\rho gz+\frac{\mu_0}{2}\Big(M(H,C)\frac{H_n}{H}\Big)^2+\mu_0H_0\int_0^H M(H,C)dH+c \quad \text{on } \Gamma.$$

Here σ is the surface tension coefficient, $\mathcal{K}$ the sum of principal curvatures, scaled over a, ρ is the fluid density, g the acceleration of gravity. The magnetic constant $\mu_0=4\pi\times10^{-7}\mathrm{H/m}$, the magnetic field H is determined from the fluid side $H=|\nabla\phi_1|$ and $H_n=\nabla\phi_1\cdot\boldsymbol{n}$, $\boldsymbol{n}$ the normal vector. The constant c is unknown. In the rfFollowing all unknown constants will be denoted by c.

We substitute the explicit formula (2.1) for the magnetisation $M(H,C)$ into the Young-Laplace equation to get on Γ

$$\mathcal{K}=-\lambda^2z+\lambda\mathrm{Si}\Big(C(\gamma H)L(\gamma H)\frac{H_n}{H}\Big)^2+\frac{2\lambda\mathrm{Si}}{3\chi}\int_0^{\gamma H}C(\gamma H)L(\gamma H)d(\gamma H)+c.$$

Here, $\lambda=a\sqrt{\rho g/\sigma}$ and $\mathrm{Si}=\mu_0M_s^2/(2\sqrt{\rho g\sigma})$ are dimensionless parameters. The integral can be computed as follows

$$\int_0^{\gamma H}C(t)L(t)\,dt=\frac{\delta}{I}\int_0^{\gamma H}\psi(t)L(t)\,dt=\frac{\delta}{I}\int\limits_0^{\gamma H}\mathrm{e}^{L^*(t)}dL^*(t)$$
$$=\frac{\delta}{I}\Big(\mathrm{e}^{L^*(\gamma H)}-1\Big)=\frac{\delta}{I}\Big(\psi(\gamma H)-1\Big),$$

where $L^*(t)=\int_0^t L(x)dx$ and $dL^*(t)=L(t)dt$. Then, the Young-Laplace equation reads:

$$\mathcal{K}=-\lambda^2z+f(H)+c \quad \text{on } \Gamma$$

with

$$f(H)=\frac{\lambda\mathrm{Si}\delta^2}{I^2}\Big(\psi(\gamma H)L(\gamma H)\frac{H_n}{H}\Big)^2+\frac{2\lambda\mathrm{Si}\delta}{3\chi I}\Big(\psi(\gamma H)-1\Big).$$

The surface Γ is described by the parametric functions $r(s)$ and $z(s)$:

$$\Gamma=\Big\{(r,z)\ \Big|\ r=r(s),\ \ z=z(s),\ \ s=[0,\ell]\Big\},$$

where the parameter s is the arc length of the free boundary Γ measured from the top of the peak ($s=0$) up to the peak foot ($s=\ell$). Space variables are dimensionless over a. Based on the approach in [10], the Young-Laplace equation is reformulated as a system of second-order ordinary differential equations for the unknown functions $\bar{r} = r/\ell$ and $\bar{z} = z/\ell$, where $\bar{s} = s/\ell \in [0,1]$

$$\begin{aligned} &\bar{r}'' = -\bar{z}'F, \quad \bar{z}'' = \bar{r}'F \qquad 0 < \bar{s} < 1; \\ &\bar{r}(0) = 0, \quad \bar{r}'(1) = 1; \quad \bar{z}'(0) = 0, \quad \bar{z}(1) = \ell^2 \int_0^1 \bar{r}^2 \bar{z}'\, d\bar{s}; \\ &F = -\frac{\bar{z}'}{\bar{r}} + \lambda^2 \ell^2 \bar{z} - \ell f(H) + 2\ell^3 \int_0^1 \bar{r}\bar{r}' f(H)\, d\bar{s}, \quad \ell = \frac{1}{\bar{r}(1)}. \end{aligned} \tag{2.9}$$

The nonlocal boundary condition is due to the integration by parts of the volume conservation condition $\int_0^1 \bar{z}\bar{r}\bar{r}'\, d\bar{s} = 0$.

The full mathematical model consists of subproblem (2.2)–(2.7) for the magnetic potential, explicit analytical formula (2.8) for the concentration and subproblem (2.9) for the equilibrium magnetic-fluid surface shape.

The three subproblems are coupled to each other. In order to determine the magnetic field structure, the knowledge about the particle distribution inside the fluid and the interface position between fluid and air is needed. The particle concentration itself depends explicitly on the magnetic field configuration. Finally, the interface between fluid and air is defined as an equilibrium shape of a free magnetic-fluid surface depending both on the magnetic field and on the concentration.

Notice that problems with nonlocal boundary conditions is of an interest in modern computational mathematics. As an important contribution to this field, we mention papers [3, 4], where finite-difference schemes for linear boundary-value problems with nonlocal boundary conditions are investigated.

3 Numerical Treatment

An iterative decoupling strategy is applied to the coupled problem (2.2)–(2.9). Three steps are performed at the n-th iteration. The first step deals with the subproblem for the magnetic potential, whereas the concentration and the interface position are unchanged and taken from the $(n-1)$-th iteration. In the second step we compute $C^n = C(\gamma H^n)$ and $H^n = |\nabla \phi_1^n|$. The third step updates the interface position Γ^n, using the last computed values for H^n and C^n. The iterations are stopped when the change in the surface shape is smaller than 10^{-6}. The initial surface configuration Γ^0 is chosen as a small perturbation of the plane surface with an amplitude of around 1% of the wavelength. We interpret a damping of the perturbation in the course of iterations as stability of the plane surface. Convergence to a solution with a curved surface indicates the onset of the Rosensweig instability. In the latter case, the initial perturbation evolves to a stationary configuration of a finite amplitude. Thereafter, Γ^0 is defined by the last computed equilibrium surface.

The magnetostatic problem (2.2)–(2.7) is solved by a finite-element method. The problem is a nonlinear elliptic second-order boundary value problem with discontinuous coefficients and mixed Dirichlet-Neumann boundary conditions. Due to its formulation in cylindrical coordinates and the assumption of axial symmetry, a variational formulation in a weighted Sobolev space $W^{1,2}(\Omega, w)$ with the weight function $w(r,z) = r$ is used. The existence of a unique weak solution was shown in [8] in case of a uniform concentration. The variational problem is discretized by continuous piecewise linear functions on triangles. Structured meshes with 160×800 nodes both inside and outside of the magnetic fluid are used. The two-dimensional meshes over the domain Ω and the one-dimensional meshes over the interface Γ are matched pointwise. Nonlinearities in the discrete equations are treated by a fixed-point iteration. The resulting system of linear equations is solved by a successive over relaxation method (SOR) in each iteration.

An iterative finite-difference scheme of the second order approximation for the parametric Young-Laplace equations was constructed in [10]. We apply the same strategy for system (2.9)

$$\frac{1}{\tau}\left(\bar{r}^{k+1}_{ss,i} - \bar{r}^{k}_{ss,i}\right) + \bar{r}^{k}_{ss,i} + \bar{z}^{k}_{\overset{\circ}{s},i} F^k_i = 0, \quad i = 1, ..., N-1,$$

$$\bar{r}^{k+1}_0 = 0, \quad \frac{\bar{r}^{k+1}_N - \bar{r}^{k+1}_{N-1}}{h} = 1, \quad \ell^k = \frac{1}{\bar{r}^k_N},$$

$$\frac{1}{\tau}\left(\bar{z}^{k+1}_{ss,i} - \bar{z}^{k}_{ss,i}\right) + \bar{z}^{k}_{ss,i} - \bar{r}^{k}_{\overset{\circ}{s},i} F^k_i = 0, \quad i = 1, \ldots, N-1,$$

$$\frac{\bar{z}^{k+1}_1 - \bar{z}^{k+1}_0}{h} = \frac{h}{2} F^k_0, \; \bar{z}^{k+1}_N = \left(\ell^k\right)^2 \sum_{i=1}^{N} \left[\left(\bar{z}^k_i - \bar{z}^k_{i-1}\right)\left(\frac{\bar{r}^k_{i-1} + \bar{r}^k_i}{2}\right)^2\right],$$

$$F^k_i = -\frac{\bar{z}^k_{\overset{\circ}{s},i}}{\bar{r}^k_i} + \lambda^2\left(\ell^k\right)^2 \bar{z}^k_i - \ell^k f(H^k_i) + \left(\ell^k\right)^3 \sum_{i=1}^{N} \left[\left(\left(\bar{r}^k_i\right)^2 - \left(\bar{r}^k_{i-1}\right)^2\right) f(H^k_i)\right].$$

Here $\{\bar{r}_i\}_{i=0}^N$ and $\{\bar{z}_i\}_{i=0}^N$ are grid-functions uniformly distributed over the free surface with a step size $h = 1/N$. The difference quotients correspond to the central derivatives ($\bar{r}_{\overset{\circ}{s}}$, $\bar{z}_{\overset{\circ}{s}}$) and the second derivatives ($\bar{r}_{ss}$, $\bar{z}_{ss}$). Nonlinearities of equations (2.9) are resolved by iterations, resulting in a three-diagonal system for the unknown grid functions at the $(k+1)$-th iteration. A relaxation technique with a parameter τ is applied to improve numerical stability.

4 Numerical Results

Numerical calculations were performed for the magnetic fluid EMG 901 (Ferrotec) with the following characteristic properties: $\chi = 2.2$, $\rho = 1406$ kg/m^3, $\sigma = 0.025$ kg/s^2, $M_s = 48$ kA/m, $C_0 = 0.107$. The control parameter of the model is a dimensionless applied field intensity $\gamma = 3\chi H_0/M_s$. All others dimensionless characteristics (χ, λ, Si) are fixed.

We set $\delta = 5$. Computations with different δ have shown that the error caused by replacing the unbounded domain by a bounded one is less than 1%.

A linear stability analysis was carried out in [13] for the Rosensweig instability under the assumption of a uniform particles distribution. The stability theory allows to get a critical value of the magnetic field intensity inside the fluid H_c as solution of the nonlinear equation

$$M(H_c)^2 = \frac{2\sqrt{\rho g \sigma}}{\mu_0}\left(1 + \left(1 + \frac{M(H_c)}{H_c}\right)^{-1}\left(1 + \frac{\partial M}{\partial H}(H_c)\right)^{-1}\right).$$

To define the critical applied field H^*, conditions (2.4), written for dimensional fields, are satisfied at the flat interface $z = 0$

$$\mu(1, H_c)H_c = H^*, \quad \text{where } \mu(1, H_c) = 1 + \frac{3\chi}{\gamma}\frac{L(\gamma H_c)}{H_c} \text{ for } \gamma = \frac{3\chi}{M_s}.$$

For the considered magnetic fluid we get $\gamma_c = 3\chi H^*/M_s = 1.252$. The stability theory predicts a critical value of the pattern wavelength $\lambda_c = 2\pi/\sqrt{\rho g/\sigma}$. The hexagonal pattern wavenumber λ_{hex} is related to the distance between two nearest peaks by the relation $\lambda_{hex} = \sqrt{3}a$. In the model we define that for any applied field intensity

$$\lambda_{hex} = \lambda_c \quad \Rightarrow \quad \lambda = a\sqrt{\rho g/\sigma} = \frac{\lambda_c}{\sqrt{3}}\sqrt{\rho g/\sigma} = \frac{2\pi}{\sqrt{3}}.$$

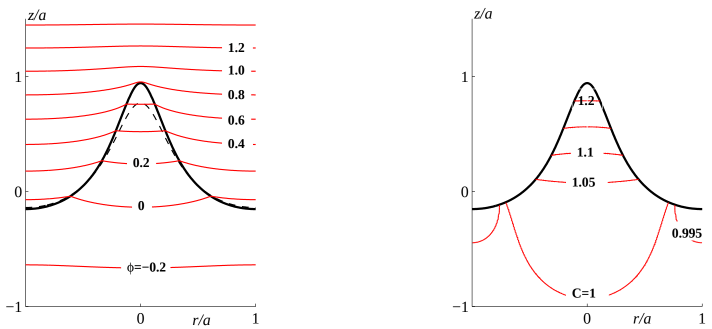

Figure 2. Overcritical free-surface shapes, isolines of the dimensionless potential (left) and of the dimensionless concentration (right) at the applied field intensity $\gamma = 1.265$. Solid (dashed) shape-lines corresponds to the non-uniform (uniform) particle distribution.

Fig. 2 (left) shows equilibrium free-surface shapes in the case of the uniform and non-uniform particle concentration for the applied field $\gamma = 1.265$. A more elongated peak region is formed for the non-homogeneous magnetic fluid. Isolines of the magnetic potential and the relative concentration in the equilibrium state are shown as well. The main inhomogeneity of the particle distribution occurs in the peak region. The concentration for a homogeneous fluid would be one in the whole volume.

The distribution of the dimensionless magnetic field intensity inside of the magnetic fluid and in the air is presented in Fig. 3. The magnetic field inside of the flat layer is uniform with the intensity $h_0^1 = 0.315$ (see (2.7)), which jumps to the value one at the fluid-air interface ($z = 0$). For equilibrium surfaces of finite amplitude the intensity inside of the fluid increases monotonically from h_0^1 at $z = -\delta$ to a value close to one at the interface. Crossing the interface, the magnetic intensity jumps to the value 3.174 and decreases monotonically up to one at $z = \delta$. The magnetic field achieves the maximum at the peak top, which is three time greater than the value taken inside the layer. The inhomogeneity of the magnetic particle distribution strengthens the magnetic field in the peak region compared to the case of a uniform particle concentration.

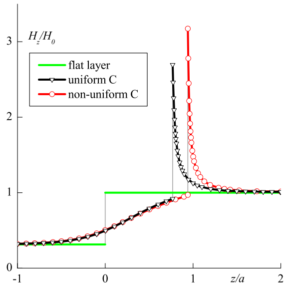

Figure 3. Distribution of the field intensity over the peak axis ($r = 0$) at $\gamma = 1.265$ in comparison with a flat layer at $\gamma < \gamma_c$.

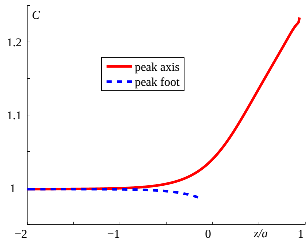

Figure 4. Distribution of the particle concentration over the peak axis ($r = 0$) and the peak foot ($r = a$) at $\gamma = 1.265$.

Fig. 4 shows the non-uniform equilibrium distribution of the particle concentration over the peak axis and the peak foot. The concentration increases monotonically in z-direction, moving along the peak axis. The relative concentration takes a value at the peak top which is about 25% greater than in the fluid bulk. The concentration increases at places with higher magnetic intensities and takes the smallest value at the peak foot. The magnetic fluid becomes uniform over the fluid volume for $z/a < -1$. The particle diffusion mechanism is absent there and the relative concentration is constant $C = 1$.

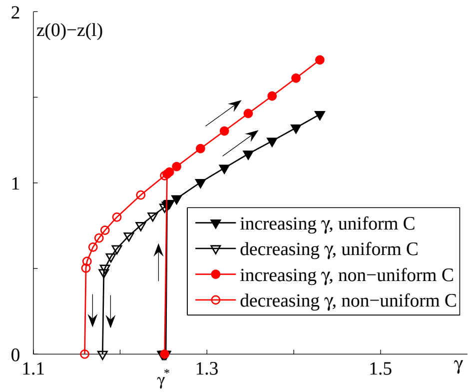

Figure 5. Dimensionless peak amplitude vs. the applied field intensity.

Fig. 5 shows the dependency of a peak amplitude on the applied field intensity γ for a uniform and non-uniform distribution of the particles. The parameter γ is increased starting from zero. The free surface remains flat until the critical point γ^* is reached. At $\gamma = \gamma^*$ the fluid configuration is modified and a formation of surface protuberances takes place. Computations show that the value of γ^* does not depend on the particle distribution and equals 1.254 ± 0.001 both for the uniform and non-uniform concentrations. A critical value $\gamma_c = 1.252$ predicted theoretically nearly coincides with the numerically obtained value γ^*. The flat surface configuration preserves the uniformity of particle distribution. That is why, the concentration effect does not influence the onset of instability. In the overcritical range $\gamma > \gamma^*$ the peak amplitude increases for the increasing γ. The non-uniform particle distribution results in a 20% higher peak amplitude in comparison to a homogeneous fluid. The amplitude difference between homogeneous and non-homogeneous fluids increases for stronger magnetic fields. If we choose any overcritical solution as an initial state and decrease γ then an effect of hysteresis takes place in the subcritical region ($\gamma < \gamma^*$). A state of an undisturbed plane surface is reached at the turning point $\gamma_* < \gamma^*$. The value of γ_* is smaller for the non-homogeneous fluid ($\gamma_* = 1.161 \pm 0.001$) than for the homogeneous one ($\gamma_* = 1.181 \pm 0.001$). We point out that two stable surface configurations are resolved in the hysteretic interval $\gamma_* < \gamma < \gamma^*$. These are the plane surface and the curved surface of a finite amplitude. Which of these two states is formed, depends on the amplitude of the initial surface perturbation Γ^0. The recent experimental [12] and numerical [9] results confirm the existence of an additional localized stable state (ferrosoliton) in the hysteretic interval.

Note that a hysteretic behavior of magnetic fluids in normal fields has been predicted theoretically in [6] and verified experimentally at first in [1].

5 Conclusions

In the paper, the mathematical model of the Rosensweig instability phenomenon, taking into consideration the magnetic particle inhomogeneity, is developed. It is based on the exact solution for the steady-state particle diffusion problem, obtained in [11]. The influence of the particle diffusion on a free magnetic-fluid surface shape for the Rosensweig instability is studied numerically for the first time. It is shown that the particle diffusion influences on overcritical free surface configurations appreciably. The nonuniform particle distribution results in about 20% higher peak amplitude in comparison to the uniform distribution approximation. At the same time, the diffusion effect does not influence on the onset of instability. The critical magnetic field intensity values coincide in homogeneous and inhomogeneous cases.

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