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On Dependence of Sets of Functions on the Mean Value of their Elements

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Abstract. The paper considers, for a given closed bounded set $M \subset \mathbb{R}^m$ and $K = (0,1)^n \subset \mathbb{R}^n$, the set $\mathcal{M} = \{h \in L_2(K;\mathbb{R}^m) \mid h(x) \in M\ a.e.\, x \in K\}$ and its subsets

$$\mathcal{M}(\hat{h}) = \Big\{h \in \mathcal{M} \mid \int_K h(x)dx = \hat{h}\Big\}.$$

It is shown that, if a sequence $\{\hat{h}_k\} \subset coM$ converges to an element $h_k \in \mathcal{M}(\hat{h}_k)$ there is $h_k^{'} \in \mathcal{M}(\hat{h}_0)$ such that $h_k^{'} - h_k \to 0$ as $k \to \infty$. If, in addition, the set M is finite or M is the convex hull of a finite set of elements, then the multivalued mapping $\hat{h} \to \mathcal{M}(\hat{h})$ is lower semicontinuous on coM.
Key words: multivalued mapping, subsets of functions with fixed mean value, continuous dependence.

1 Introduction

Most sets of admissible control functions in the theory of optimal control are given as sets of measurable functions with values from a given set: for a given reference domain $Q \subset \mathbb{R}^n$ and a given set $M \subset \mathbb{R}^m$ the set of admissible controls is defined as

$$\mathcal{M} = \Big\{h \,\text{is measurable in}\, Q \mid h(x) \in M\, a.e.\, x \in Q\Big\}.$$

Here n and m are arbitrary fixed positive integers.

Provided that Q is a bounded domain and M is a bounded and closed set, the set $\mathcal{M}$ can be split as $\mathcal{M} = \bigcup_{\hat{h} \in coM} \mathcal{M}(\hat{h})$, where

$$\mathcal{M}(\hat{h}) := \Big\{h \text{ measurable}, h(x) \in M\ a.e.\, x \in Q, \quad \frac{1}{\mid Q \mid}\int_Q h(x)\,dx = \hat{h}\Big\}.$$

Here by coA we denote the convex hull of the set A and by $\mid Q \mid$ we denote the Lebesgue measure of the set $Q \subset \mathbb{R}^n$.

Such a representation of $\mathcal{M}$ is useful when weak limits of sequences of control functions are involved, especially in procedures of relaxation via convexification, see, for instance, Warga [4]. Analogous splitting is used in the homogenization theory defining the so-called G_θ-closures, see, for instance, Milton [2]. The corresponding relaxation procedures often involve the evaluation of integrals (over the periodicity cell $K = (0,1)^n$) of the kind

$$I(\hat{h}) = \inf_{h \in \mathcal{M}(\hat{h})} \int_K f(x, h(x))\, dx \tag{1.1}$$

and the investigation of continuity properties of the function $\hat{h} \to I(\hat{h})$. To do that, obviously, one needs to know certain properties of the dependence of sets $\mathcal{M}(\hat{h})$ on $\hat{h}$.

In Sections 2 and 3 we shall show the following results.

Theorem 1. *Let $Q \subset \mathbb{R}^n$ be bounded Lipschitz domain and let the set $M \subset \mathbb{R}^m$ is bounded and closed. Then for every given sequences $\{\hat{h}_k\}$ and $\{h_k\}$ such that*

(i) $\{\hat{h}_k\} \subset coM$ and $\hat{h}_k \to \hat{h}_0$ in $\mathbb{R}^m$ as $k \to \infty$;

(ii) $h_k \in \mathcal{M}(\hat{h}_k),\ k = 1, \dots,$

there exists a sequence $\{h_{0k}\} \subset \mathcal{M}(\hat{h}_0)$ such that

$$h_k - h_{0k} \to 0 \text{ strongly in } L_2(Q; \mathbb{R}^m) \text{ as } k \to \infty.$$

Theorem 2. *Let $Q \subset \mathbb{R}^n$ be bounded Lipschitz domain and let the set $M \subset \mathbb{R}^m$ is finite or M is the closed convex hull of a finite set of elements. Then for every fixed sequence $\{\hat{h}_k\} \subset coM$ that converges in $\mathbb{R}^m$ to an element $\hat{h}_0$ and for every given element $h_0 \in \mathcal{M}(\hat{h}_0)$ there exists a sequence $\{h_k\}$, $h_k \in \mathcal{M}(\hat{h}_k)$, $k = 1, 2, \dots,$ such that*

$$h_k - h_0 \to 0 \text{ strongly in } L_2(Q; \mathbb{R}^m) \text{ as } k \to \infty.$$

Remark 1. Obviously, from Theorem 2 it follows immediately that the multivalued mapping $\hat{h} \to \mathcal{M}(\hat{h})$ is lower semicontinuous on coM (for the definition and properties of multivalued mappings we refer to Kuratowski [1]).

Remark 2. It is easy to see that under hypotheses of Theorem 1 the function $\hat{h} \to I(\hat{h})$ defined by (1.1) is lower semicontinuous provided that f is Caratheodory function and that f has a majorant $f_0 \in L_1(Q)$ (we recall that the set M is bounded). If, in addition, the hypotheses of Theorem 2 are satisfied, then the function $\hat{h} \to I(\hat{h})$ is continuous on coM.

2 Proof of Theorem 1

In this Section, we give the proof of Theorem 1. Since all reasonings below do not depend on concrete properties of the reference domain Q, then, without loosing a generality, all proofs are given for the standard case $Q = K := (0,1)^n$. Since the set M is bounded and closed, then the convex hull coM of M is closed too and all sets $\mathcal{M}(\hat{h})$ with $\hat{h} \in coM$ are nonempty closed sets. In what follows, we shall use the notion of the relative interior riA for convex sets A from Euclidean spaces, for instance $ricoM$ stands for the relative interior of the convex hull of M. For the definition of riA and other notations and properties for convex sets we refer to Rockafellar [3]. Let r_0 be dimension of coM.

Step 1. Let $\hat{h}_0 \in ricoM$. Then there exists $d > 0$ such that $\hat{h} \in ricoM$ whenever $\hat{h} \in coM$ and $|\hat{h} - \hat{h}_0| \le d$. Let us fix $\varepsilon > 0$, $0 < \varepsilon < d/4$, and let $|\hat{h} - \hat{h}_0| \le \varepsilon$. Then the element

$$\hat{h}_* = \hat{h} + \frac{d}{\varepsilon}(\hat{h}_0 - \hat{h}) \in ricoM.$$

Let $h \in \mathcal{M}(\hat{h})$, $h_* \in \mathcal{M}(\hat{h}_*)$ be arbitrary chosen elements. By virtue of Lyapunov's theorem on the range of vectorial measures for every $\lambda \in [0, 1]$ there exists a measurable set $E_\lambda \subset K$ such that

$$|E_\lambda| = \lambda, \qquad \int_{E_\lambda} h(y)\,dy + \int_{K\setminus E_\lambda} h_*(y)\,dy = \lambda\hat{h} + (1-\lambda)\hat{h}_*.$$

For a special choice $\lambda = \lambda_0 = 1 - \varepsilon/d$ we define h_0 as

$$h_0(\,\cdot\,) = \chi_{E_{\lambda_0}}(\,\cdot\,)h(\,\cdot\,) + (1 - \chi_{E_{\lambda_0}}(\,\cdot\,))h_*(\,\cdot\,),$$

where χ_E denotes the characteristic function of the set E. By construction, $h_0 \in \mathcal{M}(\hat{h}_0)$ and

$$\int_K (h(y) - h_0(y))^2\,dy = \int_{K\setminus E_{\lambda_0}} (h(y) - h_0(y))^2 dy \le 4c(M)\varepsilon/d,$$

where $c(M)$ depends only on M. Thus, the assertion of Theorem 1 holds whenever $\hat{h}_0 \in ricoM$.

Step 2. Let $\hat{h}_0$ does not belong to $ricoM$. Because $ricoM$ is not empty (provided that M consists of more than one element), then there exist a vector $a \in \mathbb{R}^m$ and a constant c such that

$$|a| = 1, \quad \langle a,\, \hat{h}_0\rangle = c < \langle a,\, \hat{h}\rangle \quad \text{for all } \hat{h} \in ricoM.$$

Without loosing generality, we can assume that $c = 0$, otherwise we can use the transform $\hat{h} \mapsto \hat{h} - \hat{h}_0$.

Let $M_1 = \big\{h \in M \,|\, \langle a,\, h\rangle = 0\big\}$. Because the sets M and M_1 are compact, then there exists a continuous function γ, $\gamma(t) = 0$ if $t \le 0$, $\gamma(t) > 0$ if $t > 0$, such that

$$\langle a,\, h - \hat{h}_0\rangle \ge \gamma(\operatorname{dist}\{h;\, M_1\}) \ \text{ for all } h \in M. \tag{2.1}$$

Without loosing generality, we can assume that the function γ is convex, otherwise we can pass to the bipolar γ^{**}, which has the desired properties. By construction, for nonnegative τ there exists the inverse function $\tau \to \gamma^{-1}(\tau)$, $\gamma^{-1}(\gamma(t)) = t$ for $t \ge 0$, which is continuous and strictly increasing on $\{\tau \in$

$\mathbb{R}|\,\tau \geq 0\}$. Now, from (2.1) and convexity of γ it follows that for every chosen $h \in \mathcal{M}$ there exists an element h_*,

$$h_* \in \mathcal{M}_1 = \Big\{ h \in L_2(K; \mathbb{R}^r) |\, h(y) \in M_1 \quad \text{a.e. } y \in K \Big\},$$

such that

$$\begin{aligned} \| \, h - h_* \, \|^2_{L_2(K;\mathbb{R}^m)} &\leq c(m, M) \int_K |h(y) - h_*(y)|\, dy \\ &\leq c(m, M)\gamma^{-1} \left(\gamma \left(\int_K |h(y) - h_*(y)|\, dy \right) \right) \\ &\leq c(m, M)\gamma^{-1} \Big(\int_K \gamma(|h(y) - h_*(y)|)\, dy \Big) \\ &\leq c(m, M)\gamma^{-1} \Big(\Big| \int_K h(y)\, dy - \int_K h_*(y)\, dy \Big| \Big), \end{aligned}$$

where $c(m, M)$ depends only on m and M. This way, for our situation with a fixed $\hat{h}_0 \in coM_1$, for every $\hat{h} \in coM$ and arbitrary chosen $h \in \mathcal{M}(\hat{h})$ there exists a corresponding $h_* \in \mathcal{M}_1$ such that

$$\| \, h - h_* \, \|^2_{L_2(K;\mathbb{R}^m)} \leq c(m, M)\gamma^{-1}(|\hat{h} - \hat{h}_0|).$$

By construction,

$$\int_K h_*(y)\, dy = \hat{h}_* \in coM_1,$$

$\mathcal{M}(\hat{h}_0) \subset \mathcal{M}_1$ and the dimension of coM_1 is less than r_0. From now on, we have to approximate the element $h_* \in \mathcal{M}(\hat{h}_*) \subset \mathcal{M}_1$ by elements from $\mathcal{M}(\hat{h}_0) \subset \mathcal{M}_1$, i.e. we have reduced the dimension r_0 of our problem to the problem with dimension less than or equal to $r_0 - 1$.

Step 3. To conclude our reasoning by induction over the dimension r_0 we have to prove our assertion for the case $r_0 = 1$. If $\hat{h}_0 \in ricoM$, then we apply reasoning from Step 1. If $\hat{h}_0$ does not belong to $ricoM$, then the set M_1 from the Step 2 consists of only one element $\hat{h}_0$ and the set $\mathcal{M}_1$ consists of one constant function $h_0(y) = \hat{h}_0$ a.e. $y \in K$. For this case we can apply the same reasoning as in Step 2, what gives the statement of Theorem for $r_0 = 1$.

3 Proof of Theorem 2

In this Section, we give the proof of Theorem 2. Let $M = \{\overline{h}_1, \ldots, \overline{h}_N\} \subset \mathbb{R}^m$. Let H be $m \times N$ matrix with columns $\overline{h}_1, \ldots, \overline{h}_N$ respectively and let

$$\Lambda := \Big\{ \overline{\lambda} \in \mathbb{R}^N \mid \overline{\lambda} = (\lambda_1, \ldots, \lambda_N)\,, \lambda_j \geq 0,\, j = 1, \ldots, N;\, \lambda_1 + \cdots + \lambda_N = 1 \Big\}.$$

To a given vector-function $h \in \mathcal{M}$ (it has only N admissible values from M) we can appoint an element $\overline{\lambda}$ whose components λ_j represent the volume fractions in K of the sets where the vector-function h has the value $\overline{h}_j$, $j = 1, \dots, N$, respectively. Let $E := \{\overline{z} \in \mathbb{R}^N \mid H\overline{z} = 0\}$.

In these notations the statement of Theorem 2 is a straight consequence of:

$$\begin{cases} if\, \hat{\lambda}_0 \in \Lambda,\, \{\overline{a}_k\} \subset \mathbb{R}^N,\, \overline{a}_k \to 0 \,\text{as}\, k \to \infty \\ \qquad \text{and}\, \{\hat{\lambda}_0 + \overline{a}_k + E\} \bigcap \Lambda \neq \emptyset,\ k = 1, 2, \dots, \\ \text{then there exists a sequence}\, \{\overline{\lambda}_k\}\, \text{such that} \\ \overline{\lambda}_k \to \hat{\lambda}_0 \,\text{as}\, k \to \infty, \\ \overline{\lambda}_k \in \{\hat{\lambda}_0 + \overline{a}_k + E\} \bigcap \Lambda,\ k = 1, 2, \dots . \end{cases} \tag{3.1}$$

Indeed, first of all we have to take care only about volume fractions of sets where the functions under consideration take the corresponding values $\overline{h}_1, \dots, \overline{h}_N$ (we always can prearrange the corresponding sets preserving their measures). Further, for $h \in \mathcal{M}$ with corresponding volume fractions $(\lambda_1, \dots, \lambda_N) = \overline{\lambda}$ we have that $h \in \mathcal{M}(H\overline{\lambda})$, and every $\hat{h} \in coM$ has the representation $\hat{h} = H(\hat{\lambda} + E)$ with some $\hat{\lambda} \in \Lambda$. The convergence $\overline{a}_k \to 0$ as $k \to \infty$ in (3.1) implies the corresponding convergence $\hat{h}_k \to \hat{h}_0$ in Theorem 2, and the convergence $\overline{\lambda}_k \to \hat{\lambda}_0$ implies the corresponding convergence $h_k \to h_0$ in Theorem 2.

Let us denote $\overline{1} = (1, \dots, 1) \in \mathbb{R}^N$ and let us represent E as the direct sum $E = E_0 \oplus E_1$ where

$$E_0 = \{\overline{z} \in E \mid \langle \overline{z}, \overline{1} \rangle = 0\}.$$

Here the subspace E_1 can be equal to $\{0\}$ if $\overline{1}$ is orthogonal to E. From assumptions on $\overline{a}_k$ we have the existence of $\overline{z}_{0k} \in E_0$ and $\overline{z}_{1k} \in E_1$ such that

$$\hat{\lambda}_0 + \overline{a}_k = \overline{z}_{0k} + \overline{z}_{1k} \in \Lambda,$$
$$\langle \overline{a}_k + \overline{z}_{1k}, \overline{1} \rangle = 0, \quad \overline{z}_{1k} \to 0 \quad \text{as} \quad k \to \infty.$$

So, if necessary, using the transform $\overline{a}_k \to \overline{a}_k + \overline{z}_{1k}$ and replacing E by E_0, without loosing generality, we can assume that

(i) the vector $\overline{1}$ is orthogonal to E;

(ii) $\langle \overline{a}_k, \overline{1} \rangle = 0$, $k = 1, 2, \dots$.

That means (since $\langle \hat{\lambda}_0 , \overline{1} \rangle = 1$) our further reasoning concerns only the hyperplane $\{\overline{z} \in \mathbb{R}^N \mid \langle \overline{1}, \overline{z} \rangle = 1\}$. There are two possibilities:

(a) $(\hat{\lambda}_0 + E) \bigcap ri\Lambda \neq \emptyset$;

(b) $\hat{\lambda}_0$ belongs to a façade Λ_s of Λ with the dimension s, $0 \le s \le N - 2$, and $\hat{\lambda}_0 + E$ can be separated from $ri\Lambda$.

For the case (a) there exists a $\lambda_* \in (\hat{\lambda}_0 + E) \bigcap ri\Lambda$ and the elements

$$\lambda_k = \hat{\lambda}_0 + \overline{a}_k + \tau_k(\overline{\lambda}_* - \hat{\lambda}_0), \quad k = 1, 2, \dots,$$

with appropriate $\tau_k > 0$, $k = 1, 2, \dots$, solve the problem for $k \ge k_0$ with some k_0.

For the case (b), without loosing generality, we can assume that Λ_s is the façade with the minimal dimension s compared to all façades, which contain $\hat{\lambda}_0$. Hence, after relabeling indexes we obtain

$$\Lambda_s = \Big\{ \overline{\lambda} \in \Lambda \mid \overline{\lambda} = (\lambda_1, \dots, \lambda_N)\,,\, \lambda_{s+2} = \cdots = \lambda_N = 0 \Big\},$$
$$\hat{\lambda}_0 = \Big(\lambda_1^0, \dots, \lambda_N^0\Big), \quad 0 < \lambda_1^0, \dots, 0 < \lambda_{s+1}^0, \quad \lambda_{s+2}^0 = \cdots = \lambda_N^0 = 0.$$

If

$$\hat{\lambda}_0 + \overline{a}_k + \overline{z}_k \in \Lambda \quad \& \quad \overline{z}_k \in E\,,\; k = 1, 2, \dots,$$

then from

$$\overline{a}_k \to 0 \quad \text{as} \quad k \to \infty, \quad \text{and} \quad \langle \overline{z}_k, \overline{1} \rangle = 0\,, k = 1, 2, \dots,$$

it follows immediately that the sequence $\{\overline{z}_k\}$ is bounded.

Let us assume that the sequence $\{\overline{z}_k\}$ converges to an element $\overline{z}_0$. If $\overline{z}_0 = 0$, then the sequence

$$\overline{\lambda}_k := \hat{\lambda}_0 + \overline{a}_k + \overline{z}_k,\; k = 1, 2, \dots,$$

solves the problem.

If $\overline{z}_0 \neq 0$, then those entries of $\overline{z}_0 = (z_{01}, \dots, z_{0N})$, which are different from zero, are positive for $j \geq s+2$ and negative for those indexes $j^{''}$, for which $\lambda_{j''}^0 = 1$ (if any). Therefore, there exists $d_0 > 0$ such that $\hat{\lambda}_0 + \tau\overline{z}_0 \in \Lambda$ provided $0 \leq \tau \leq d_0$.

Since $\overline{z}_k - \overline{z}_0 \to 0$ as $k \to \infty$, then the elements

$$\overline{\lambda}_k := \hat{\lambda}_0 + \tau_k \overline{z}_0 + (\overline{z}_k - \overline{z}_0) + \overline{a}_k$$

for $k \geq k_0$ and with appropriate τ_k, $\tau_k \to 0$ as $k \to \infty$, belong to Λ. Indeed, since $\langle \overline{z}_0\,, \overline{1} \rangle = 0\,, \langle \overline{a}_k\,, \overline{1} \rangle = 0\,, \langle \overline{z}_k\,, \overline{1} \rangle = 0\,, k = 1, 2, \dots,$ we have to check only inequalities

$$\lambda_{kj} \geq 0, \quad j = 1, \dots, N;, \quad k = k_0, k_0 + 1, \dots$$

(obviously, $\overline{\lambda}_k \to \hat{\lambda}_0$ as $k \to \infty$). For those indexes $\{j^{'}\}$, for which entries of $\overline{z}_0$ are equal to zero,

$$\lambda_{j'}^0 + a_{kj'} + (z_{kj'} - z_{0j'}) \geq 0, \quad k = 1, 2, \dots,$$

(by the initial assumptions on the sequence $\{\overline{a}_k\}$), but for the rest of indexes $\{j^{''}\}$ either

$$1 > \lambda_{j''}^0 > 0$$

or

$$\lambda_{j''}^0 = 1 \quad \& \quad z_{0j''} < 0$$

what is sufficient for the existence of τ_k with desired properties.

The general case of an arbitrary sequence $\{\overline{z}_k\}$ is treated by standard reasoning by contradiction, i.e., we assume the contrary that there exist $d > 0$ and a sequence of indexes $\{k^{'}\}$ such that the distance from $\hat{\lambda}_0$ to $\{\hat{\lambda}_0 + \overline{a}_{k'} + E\} \bigcap \Lambda$

is greater than d. After that we take an arbitrary subsequence of $\{\overline{a}_{k'}\}$, for which the corresponding sequence $\{\overline{z}_{k'}\}$ converges. The proof of the first part of Theorem 2 is completed.

Now, let M be closed convex hull of a finite number of elements $\{h_1,\ldots,h_N\}$ and let

$$S := \{\sigma \in L_2(K;\mathbb{R}^N) \mid \sigma = (\sigma_1,\ldots,\sigma_N),\, 0 \le \sigma_j(x) \le 1,\, j=1,\ldots,N; \\ \sum_{j=1}^{N} \sigma_j(x) = 1 \, a.e. \, x \in K\}.$$

Since the function

$$(\sigma, x) \to (h(x) - \sum_{j=1}^{N} \sigma_j h_j)^2$$

is a normal integrand on $\Lambda \times K$ (for every fixed $h \in \mathcal{M}$), then every $h \in \mathcal{M}$ has the representation

$$h(x) = \sum_{j=1}^{N} \sigma_j(x) h_j \quad a.e. \quad x \in K$$

with some $\sigma \in S$. In turn, a subset of piecewise constant elements is dense in S and sets $\mathcal{M}(\hat{h})$ have the same property.

This way, by using Cantor's diagonal process, we have that it is sufficient to show the existence of the approximating sequence $\{h_k\}$ for the case of a piecewise element $h_0 \in \mathcal{M}(\hat{h}_0)$. Let $Q_i \subset K,\ i=1,\ldots,s$, are the sets where the function h_0 is constant and takes values $g_1,\ldots,g_s$ from M respectively. Now, we define the set $\tilde{M} := \{h_1,\ldots,h_N,g_1,\ldots,g_s\}$ and sets

$$\tilde{\mathcal{M}}(\hat{h}) := \ \{h \,\text{measurable in}\, K \mid h(x) \in \tilde{M} \ \text{ a.e. } \ x \in K, \ \int_K h(x)dx = \hat{h}\}.$$

By construction, $co\tilde{M} = M$ and $\tilde{\mathcal{M}}(\hat{h}) \subset \mathcal{M}(\hat{h})\ \forall \hat{h} \in M$.

If $\{\hat{h}_k\} \subset M$ and $\hat{h}_k \to \hat{h}_0$ as $k \to \infty$ then also $\{\hat{h}_k\} \subset co\tilde{M}$, $\hat{h}_0 \in co\tilde{M}$ and $h_0 \in \tilde{\mathcal{M}}(\hat{h}_0)$. This way, the existence of the desired approximating sequence $\{h_k\}$ follows immediately from the proof of the first part of Theorem 2. The proof of Theorem 2 is completed.

We conclude with a simple example, which shows that the statement of Theorem 2 is not, in general, true under hypotheses of Theorem 1. Let

$$M = \Big\{(-1,0,0),\, (1,0,0),\, (0,t,t^2),\ 0 \le t \le 1\Big\} \subset \mathbb{R}^3.$$

By construction,

$$\mathcal{M}((0,t,t^2)) = \Big\{(h_1,h_2,h_3) \in L_2(K;\mathbb{R}^3) \mid \\ h_1(x) = 0,\, h_2(x) = t,\, h_3(x) = t^2;\, x \in K\Big\} \quad \text{for} \quad 0 < t < 1,$$

$$\mathcal{M}((0,0,0)) = \Big\{(h_1,h_2,h_3) \in L_2(K;\mathbb{R}^3) \mid h_1(x) = -1 \quad \text{or} \quad 0 \quad \text{or } 1, \\ \int_K h_1(x)\,dx = 0;\ h_2(x) = 0,\, h_3(x) = 0;\, x \in K\Big\},$$

and the statement of Theorem 2 does not hold.

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