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Optimal Systems and Group Invariant Solutions for a Model Arising in Financial Mathematics

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Abstract. We consider a bond-pricing model described in terms of partial differential equations (PDEs). Classical Lie point symmetry analysis of the considered PDEs resulted in a number of point symmetries being admitted. The one-dimensional optimal system of subalgebras is constructed. Following the symmetry reductions, we determine the group-invariant solutions.

Key words: point symmetries, optimal systems, bond-pricing model, invariant solutions.

1 Introduction

Over the last few decades, there has been a great interest in the modelling and analysis of problems arising in finance markets. Some of these problems are modelled in terms of PDEs. A zero-coupon bond is a contract that pays a known fixed amount at some expiry time T. Thus its pricing equation is an example of a model which can be used to evaluate interest rate derivatives. A number of studies have been devoted to the use of symmetry techniques for PDEs arising in the field of finance mathematics (see e.g. [4, 8, 12, 14]). The theory and applications of symmetries may be found in excellent texts such as [3, 7, 10, 15]. Ibragimov and Gazizov [8] have considered and analyzed the classical Black-Sholes-Merton model.

It can be shown that a stochastic process describing the spot rate x,

$$dx(t) = \mu x(t)dt + \sigma x dZ(t),$$

leads to the Black-Scholes model given by

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0,$$

where $\{u(x,t),\ t \geq 0\}$ is the asset price, μ is the measure of the average rate of growth of the asset price, σ is the volatility of the underlying asset, t is time, r is the risk free interest rate, σ^2 is the variance, $\{Z(t),\ t \geq 0\}$ is the Wiener process and $dZ(t)$ is its increment. Here r, μ and σ are constants.

Pooe *et al.* [12], assumed that the spot rate follows the stochastic process (see also [4, 5]) given by

$$dx(t) = a(x,t)\,dt + w(x,t)\,dZ(t),$$

and ended up solving the model

$$u_t + \frac{1}{2}w^2 u_{xx} + (a - \lambda(x,t)w)u_x - xu = 0.$$

Here $\lambda(x,t)$ is the market price of risk, $a(x,t)$ and $w(x,t)$ represent the expected rate of return and volatility, respectively. The value of interest rate derivatives such as bonds, swaps, naturally depends on the interest rates. The choice of the coefficients $u(x,t)$ and $w(x,t)$ is important for subsequent modelling of bond prices.

Sinkala *et al.* [14] assumed that the spot rate follows the stochastic process

$$dx(t) = \kappa(\theta - x(t))\,dt + \sigma\sqrt{x}\,dZ(t) \tag{1.1}$$

and this gave rise to the model

$$u_t + \frac{1}{2}\sigma^2 x u_{xx} + \kappa(\theta - x)u_x - xu = 0,$$

where κ and θ are constants. The stochastic process in equation (1.1) is referred to as the square root process [14].

The Ornstein-Uhlenbeck process

$$dx(t) = \kappa(\theta - x(t))\,dt + \sigma\,dZ(t), \tag{1.2}$$

leads to the asset price satisfying the PDE

$$u_t + \frac{1}{2}\sigma^2 u_{xx} + \kappa(\theta - x)u_x - xu = 0,$$

which was coupled with equation (1.2). Such a system was solved subject to the terminal condition $u(x,T) = 1$.

As a synthesis of all these equations Mahomed [9] developed a method for solving the general linear $(1+1)$ parabolic equation

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u.$$

In essence a number of rate models which led to analytical solutions have been used and more models yet to be solved. The aim of this paper is to analyze a bond-pricing model and to determine its closed-form solutions using Lie point symmetry techniques. In the present work we assume that the risk free spot rate follows the Itô's process or stochastic process of the form

$$dx(t) = b(x(t),t)w^2(x(t),t)\,dt + w(x(t),t)\,dZ(t),$$

with specified drift term $b(x(t), t)$ and the volatility $w(x(t), t)$. We first derive the governing equation and determine its Lie point symmetries in Section 2. In Section 3 we adopt methods of [10] to construct the one-dimensional optimal systems of subalgebras. The symmetry reductions and construction of group-invariant solutions are provided in Section 4 and lastly the concluding remarks are given.

2 Governing Equations and Symmetry Analysis

We adapt the following definition and lemma from [16, 17].

If $\{x(t),\, t \geq 0\}$ is a stochastic process, $\{Z(x(t), t),\, t \geq 0\}$ the Wiener process and a and b are smooth functions, then an equation of the form

$$dx(t) = a(x(t), t)\, dt + b(x(t), t)\, dZ(t)$$

is called a stochastic differential equation for which the solution is called the Itô's process.

In this paper, we follow [4] and assume that the risk free spot rate x follows the stochastic process

$$\begin{aligned} dx(t) &= [v(x(t), t) - \lambda(x(t), t)w(x(t), t)]dt + w(x(t), t)\, dZ(t) \\ &= b(x(t), t)w(x(t), t)^2 dt + w(x(t), t)\, dZ(t) \end{aligned} \tag{2.1}$$

and hence the PDE corresponding to (2.1) is given by

$$u_{xx} + \frac{2}{w^2}u_t + b(x, t)u_x - \frac{2x}{w^2}\, u = 0. \tag{2.2}$$

Considering the power law volatility $w(x, t) = cx^{3/2}$ and the nonlinear drift term $b(x, t) = \dfrac{3}{4x} - \dfrac{q}{x^{3/2}}$, we note that the risk free spot rate x follows the stochastic process

$$dx = \left[\frac{3}{4}x^2 - qx^{3/2}\right]c^2\, dt + cx^{3/2}\, dZ(t)$$

and we may rewrite equation (2.2) as

$$u_{xx} + \frac{2}{c^2x^3}u_t + 2\left(\frac{3}{4x} - \frac{q}{x^{3/2}}\right)u_x - \frac{2}{c^2x^2}u = 0. \tag{2.3}$$

Here c and q are constants. The power law volatility conforms to actual data [4]. In fact, the volatility $x^{3/2}$ has shown to be the best-fit power law [2]. Most models use linear drift (which are rejected by Aït-Sahalia [2]) and in this paper we have chosen a nonlinear drift term. Using the computer subprogram Dimsym [13] of Reduce [6], we obtain other than the infinite symmetry generator, with the base vectors

$$X_1 = \frac{(c^4q^2x^{3/2}t^2 - c^2tx^{3/2} + 4\sqrt{x} - 4c^2qtx)u}{2x^{3/2}c^2}\frac{\partial}{\partial u} + t^2\frac{\partial}{\partial t} - 2tx\frac{\partial}{\partial x},$$

$$X_2 = \frac{(c^2q^2t\sqrt{x} - 2q)u}{2\sqrt{x}}\frac{\partial}{\partial u} + t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x},$$

$$X_3 = \frac{\partial}{\partial t}, \quad X_4 = u\frac{\partial}{\partial u},$$

which span the four dimensional Lie symmetry algebra (see also [4]).

The main purpose for finding symmetries is to generate or construct invariant solutions. Note that any linear combination of the above generators may lead to the construction of group invariant solutions. In order to ensure a minimal set of reductions that are not equivalent by any transformation one may construct the one dimensional optimal system (see e.g. [10]).

3 One-Dimensional Optimal System of Subalgebras

It is well known that reduction of the independent variables by one is possible using any linear combination of our generators of symmetry X_i, $i = 1, \dots, 4$. We construct a set of minimal combinations known as optimal systems [10]. To construct the optimal systems we need the commutators of the admitted symmetries given in Table 1.

Table 1. Lie bracket of the admitted symmetry algebra, $[X_i, X_j] = X_iX_j - X_jX_i$

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	$-X_1$	$\frac{1}{2}X_4 - X_2$	0
X_2	X_1	0	$-\left(X_3 + \frac{1}{2}c^2q^2X_4\right)$	0
X_3	$2X_2 - \frac{1}{2}X_4$	$X_3 + \frac{1}{2}c^2q^2X_4$	0	0
X_4	0	0	0	0

An optimal system of a Lie algebra is a set of $l-$dimensional subalgebras such that every $l-$dimensional subalgebra is equivalent to a unique element of the set under some element of the adjoint representation [10];

$$\mathrm{Ad}(\exp(\epsilon X_i))X_j = \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}(\mathrm{ad}X_i)^n X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2!}[X_i, [X_i, X_j]] - \dots \tag{3.1}$$

The adjoint representation is constructed using the formula (3.1) and is given in Table 2.

Let us consider the linear combination of the symmetry generators:

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4. \tag{3.2}$$

Now, if we let $a_1 \neq 0$ in (3.2), one may rescale a_1 such that $a_1 = 1$. Acting on X by $\mathrm{Ad}\left(\exp\left(\kappa X_3\right)\right)$, with κ being the root of the quadratic equation

$$\kappa^2 - a_2\kappa + a_3 = 0,$$

Table 2. Adjoint representation table. At position (i,j) we have $\mathrm{Ad}(\exp(\epsilon X_i))X_j$ for $i=1,\ldots,4$.

Ad	X_1	X_2	X_3	X_4
X_1	X_1	$X_2+\epsilon X_1$	$X_3-\epsilon\left(\frac{1}{2}X_4-2X_2\right)$ $+\epsilon^2 X_1$	X_4
X_2	$\mathrm{e}^{-\epsilon}X_1$	X_2	$\mathrm{e}^{\epsilon}X_3+\frac{1}{2}c^2q^2(1-\mathrm{e}^{\epsilon})X_4$	X_4
X_3	$X_1-\epsilon(2X_2-\frac{1}{2}X_2)$ $\epsilon^2(X_3+\frac{1}{2}c^2q^2X_4)$	$X_2-\epsilon(X_3+\frac{1}{2}c^2q^2X_4)$	X_3	X_4
X_4	X_1	X_2	X_3	X_4

we obtain $X^{[i]}=X_1+a_2'X_2+a_4'X_4$. Here the coefficients a_2' and a_4' are given by

$$a_2'=a_2-2\kappa,\quad a_4'=\left(\frac{1}{2}c^2q^2\right)(\kappa^2-\kappa)+\left(\frac{\kappa}{4}+a_4\right).$$

Thus the one-dimensional subalgebra spanned by X with $a_1\neq 0$ is equivalent to the one spanned by $X_1+\alpha X_2+\beta X_4,\ \alpha,\beta\in\mathbb{R}$. Assuming $a_1=0$, $a_2\neq 0$ and setting $a_2=1$, we get $X^{[i]}=X_2+a_3X_3+a_4X_4$. Acting on $X^{[i]}$ by $\mathrm{Ad}\left(\exp\left(-\dfrac{2a_4}{c^2q^2}X_1\right)\right)$ we have $X^{[ii]}=X_2+a_3'X_3$, where $a_3'=a_3+\dfrac{2a_4}{c^2q^2}$. No further simplification is possible. In other words, every one-dimensional subalgebra generated by X with $a_2\neq 0$ is equivalent to the subalgebra spanned by $X_2+\alpha X_3,\ \alpha\in\mathbb{R}$.

Assume now that $a_2=0$ and $a_3\neq 0$. Set $a_3=1$. Then this leads to an irreducible one-dimensional subalgebra $X^{[iii]}=X_3+a_4X_4$. Thus the one-dimensional subalgebra spanned by X with $a_3\neq 0$ is equivalent to the one spanned by either $X_3+\alpha X_4,\ \ \alpha\in\mathbb{R}$. The last subalgebra is obtained by setting $a_3=0$ and $a_4=1$. In this case we have X_4. Hence the set of one-dimensional optimal systems is

$$\Big\{X_1+\alpha X_2+\beta X_4;\ X_2+\alpha X_3;\ X_3+\alpha X_4;\ X_4\Big\}.$$

4 Symmetry Reductions and Invariant Solutions

Using the members of the constructed optimal systems we perform some reductions and wherever possible solve the equations completely.

Example 1. Considering $X=X_1+\alpha X_2+\beta X_4$, we obtain the invariant $\rho=x(t^2+\alpha t)$ and the functional form of the group invariant solution

$$u=\exp\Big(\frac{c^2q^2}{2}t-\frac{1}{2}\ln(t+\alpha)+\frac{2}{c^2x(t+\alpha)}-\frac{2q}{\sqrt{x}}+\frac{\beta}{\alpha}\ln\Big(\frac{t}{t+\alpha}\Big)\Big)f(\rho),$$

where f satisfies the ordinary differential equation (ODE)

$$\rho^3f''+\Big(\frac{2\alpha}{c^2}\rho+\frac{3}{2}\rho^2\Big)f'+\Big(\frac{2}{c^2}\beta-\frac{2}{c^2}\rho\Big)f=0. \tag{4.1}$$

If we make the appropriate transformation (see e.g. [11]), for example, let $f = \rho^{-\frac{\beta}{\alpha}} v$, then (4.1) becomes

$$\rho^2 v'' + \Big[\Big(\frac{3}{2} - \frac{2\beta}{\alpha}\Big)\rho + \frac{2\alpha}{c^2}\Big]v' + \Big[-\frac{\beta}{\alpha}\Big(\frac{1}{2} - \frac{\beta}{\alpha}\Big) - \frac{2}{c^2}\Big]v = 0. \tag{4.2}$$

A further change of variables, for example, $\rho = \dfrac{1}{\xi}$, and $v = \xi^k e^{\xi} w$, where k satisfies the polynomial equation

$$k^2 - \left(\frac{1}{2} + \frac{2\beta}{\alpha}\right)k + \left(-\frac{\beta}{2\alpha} + \frac{\beta^2}{\alpha^2} - \frac{2}{c^2}\right) = 0$$

reduces equation(4.2) to

$$\xi w'' + \Big[\Big(2 - \frac{2\alpha}{c^2}\Big)\xi + 2k + \frac{1}{2} - \frac{2\beta}{\alpha}\Big]w' + \Big[\Big(1 - \frac{2\alpha}{c^2}\Big)\xi + 2k + 2 - \Big(\frac{3}{2} - \frac{2\beta}{\alpha}\Big) - \frac{2\alpha}{c^2}k\Big]w = 0. \tag{4.3}$$

Now for simplicity let

$$\tilde{a} = \left(2 - \frac{2\alpha}{c^2}\right), \quad \tilde{b} = 2k + \frac{1}{2} - \frac{2\beta}{\alpha}, \quad \tilde{c} = \Big(1 - \frac{2\alpha}{c^2}\Big), \quad \tilde{d} = 2k + 2 - \Big(\frac{3}{2} - \frac{2\beta}{\alpha}\Big) - \frac{2\alpha}{c^2}k,$$

then (4.3) can be written as

$$\xi w'' + (\tilde{a}\xi + \tilde{b})w' + (\tilde{c}\xi + \tilde{d})w = 0.$$

The latter ODE admits the family of solutions

$$w(\xi) = \xi^{-\tilde{b}/2} e^{-\tilde{a}\xi/2} \left\{k_1\, M_{\gamma,\varpi}\left(\sqrt{\tilde{a}^2 - 4\tilde{c}}\;\xi\right) + k_2\, W_{\gamma,\varpi}\left(\sqrt{\tilde{a}^2 - 4\tilde{c}}\;\xi\right)\right\},$$

where k_1, k_2 are arbitrary constants, $M_{\gamma,\varpi}(\cdot)$, $W_{\gamma,\varpi}(\cdot)$ are the Whittaker's functions (see e.g., [1]), $\gamma = (2\tilde{d} - \tilde{a}\tilde{b})/(2\sqrt{\tilde{a}^2 - 4\tilde{c}})$ and $\varpi = (\tilde{b} - 1)/2$. In terms of v and ρ we have

$$v(\rho) = \rho^{-(k + \tilde{b}/2)} e^{(2-\tilde{a})/2\rho} \Big\{k_1\, M_{\gamma,\varpi}\Big(\frac{\sqrt{\tilde{a}^2 - 4\tilde{c}}}{\rho}\Big) + k_2\, W_{\gamma,\varpi}\Big(\frac{\sqrt{\tilde{a}^2 - 4\tilde{c}}}{\rho}\Big)\Big\}.$$

In terms of the original variables, the general solution for equation (2.3) is given by

$$u(x,t) = \Big\{\exp\Big(\frac{c^2 q^2}{2}t - \frac{\ln(t+\alpha)}{2} + \frac{2}{c^2 x(t+\alpha)} - \frac{2q}{\sqrt{x}} + \frac{\beta}{\alpha}\ln\Big(\frac{t}{t+\alpha}\Big) + \frac{2-\tilde{a}}{2x(t^2+\alpha t)}\Big)\Big\} \times\; x(t^2 + \alpha t)^{-\tilde{b}/2} \Big\{k_1\, M_{\gamma,\varpi}\Big(\frac{\sqrt{\tilde{a}^2 - 4\tilde{c}}}{x(t^2 + \alpha t)}\Big) + k_2\, W_{\gamma,\varpi}\Big(\frac{\sqrt{\tilde{a}^2 - 4\tilde{c}}}{x(t^2 + \alpha t)}\Big)\Big\}.$$

Example 2. Consider $X = X_2 + \alpha X_3$ given by

$$X = \frac{(c^2q^2t\sqrt{x} - 2q)u}{2\sqrt{x}}\frac{\partial}{\partial u} + (t+\alpha)\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}. \tag{4.4}$$

The associated Lagrange's system

$$\frac{dt}{t+\alpha} = \frac{dx}{-x} = \frac{2\sqrt{x}du}{u(c^2q^2t\sqrt{x} - 2q)},$$

arising from (4.4) gives the following functional form

$$u = (t+\alpha)^{-0.5c^2q^2\alpha}\exp\Big(\frac{-c^2q^2\alpha}{2}t - \frac{2q}{\sqrt{x}}\Big)f(\rho),$$

where $\rho = x(t+\alpha)$ and f satisfies the ODE

$$\rho^3 f'' + \Big(\frac{3}{2}\rho^2 + \frac{2}{c^2}\rho\Big)f' - \Big(\alpha q^2 + \frac{2}{c^2}\rho\Big)f = 0.$$

Hence

$$f = \rho^\nu\Big\{k_1 M\Big(m, n, \frac{2}{c^2\rho}\Big) + k_2 U\Big(m, n, \frac{2}{c^2\rho}\Big)\Big\},$$

where

$$m = \frac{1+\sqrt{c^2+32}}{2c}, \quad n = \frac{2c+\sqrt{c^2+32}}{2c}, \quad \nu = -\frac{c+\sqrt{c^2+32}}{4c}$$

and k_1, k_2 are arbitrary constants. $M(a,b,\cdot)$ and $U(a,b,\cdot)$ are Kummer M and Kummer U special functions (see e.g. [1]). In terms of original variables, we obtain the group invariant solution

$$u(x,t) = (t+\alpha)^{(c^2q^2)/2}\exp\Big(-\frac{c^2q^2\alpha}{2}t - \frac{2q}{\sqrt{x}}\Big)\, x(t+\alpha)^\nu$$
$$\times \Big[k_1\, M\Big(m, n, \frac{2}{c^2x(t+\alpha)}\Big) + k_2\, U\Big(m, n, \frac{2}{c^2x(t+\alpha)}\Big)\Big].$$

Example 3. Consider $X = X_3 + \alpha X_4$. This leads to the functional form $u = e^{\alpha t}f(x)$, where $f(x)$ satisfies the ODE

$$x^3 f'' + \Big(\frac{3}{2}x^2 - 2qx^{\frac{3}{2}}\Big)f' + \frac{2}{c^2}\Big(\alpha - x\Big)f = 0.$$

This equation is similar to (4.1). Note that reduction by X_4 does not yield any solution.

5 Concluding Remark

We have considered PDEs describing bond-pricing. Symmetry analysis revealed a rich array of Lie point symmetries being admitted (see also [4]). The one-dimensional optimal systems of subalgebras are constructed and the group-invariant solutions are obtained. We have utilized the realistic power law model for the volatility and the nonlinear drift term. The PDEs associated with finance are rarely solvable and usually approximations and Monte Carlo methods are applied. However, with given realistic choices of volatility and risk-free drift term we have constructed the nontrivial close-form solutions.

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