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On Asymptotic Classification of Solutions to Fourth-Order Differential Equations with Singular Power Nonlinearity

Irina Astashova

Lomonosov Moscow State University,
Faculty of Mechanics and Mathematics
Leninskie Gory, 1, Moscow, RU 119991
Plekhanov Russian University of Economics (PRUE),
Faculty of Mathematical Economics, Statistics and Informatics (MESI)
Stremyanny lane, 36 Moscow, RU 117997
E-mail: ast@diffiety.ac.ru

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Abstract. The asymptotic behavior of all solutions to the fourth-order Emden–Fowler type differential equation with singular nonlinearity is investigated. The equation is transformed into a system on the three-dimensional sphere. By investigation of the asymptotic behavior of all possible trajectories of this system an asymptotic classification of all solutions to the equation is obtained.

Keywords: singular nonlinear equations, asymptotic classification, oscillation.

AMS Subject Classification: 34C.

1 Introduction

Classification of all possible solutions to nonlinear equations is one of the major problems in qualitative theory of ordinary and partial differential equations. There are no general methods for investigation of qualitative properties of solutions to nonlinear differential equations. In this paper the asymptotic behavior of all solutions to the fourth-order ordinary differential equation with singular power nonlinearity is investigated.

Asymptotic classifications of solutions to the fourth-order Emden–Fowler type differential equations

$$y^{(\mathrm{IV})}(x) + p_0\,|y|^k \operatorname{sgn} y = 0, \quad 0 < k < 1, \quad p_0 = \text{const} > 0, \tag{1.1}$$

$$y^{(\mathrm{IV})}(x) - p_0\,|y|^k \operatorname{sgn} y = 0, \quad 0 < k < 1, \quad p_0 = \text{const} > 0 \tag{1.2}$$

are presented.

For the regular case, $k > 1$, asymptotic classifications of all solutions to such equations of orders three and four are presented in [3], [4], [6]. In [5] asymptotic properties of solutions to the third-order equations with singular nonlinearity, $0 < k < 1$, are described. Oscillatory problems for fourth-order equations are investigated in [1], [2], [7], [8], [10], [11], [12], [13], [14], [15].

In the case of regular nonlinearity, $k > 1$, only maximally extended solutions are usually considered, because solutions can behave in a special way only near the boundaries of their domains. If $k < 1$, then special behavior can occur also near internal points of the domains. This is why a notion of *maximally unique* (MU) solutions is introduced.

Definition 1. A solution $u : (a, b) \to \mathbb{R}$ with $-\infty \le a < b \le +\infty$ to any ordinary differential equation is called a MU-solution if the following conditions hold: (i) the equation has no other solution equal to u on some subinterval of (a, b); (ii) either there is no solution defined on another interval containing (a, b) and equal to u on (a, b), or there exist at least two such solutions not equal to each other at points arbitrary close to the boundary of (a, b).

Remark 1. If an equation has a locally unique solution to any initial value problem, then the set of all its MU-solutions coincides with the set of all its maximally extended solutions. They satisfy (i) due to the uniqueness property and the first sub-condition of (ii) since they are maximally extended.

Remark 2. Consider the equation

$$u''(x) = 6\sqrt[3]{u(x)}. \tag{1.3}$$

1) The function $u(x) = x^3$ defined on $\mathbb{R}$ is a solution to equation (1.3) but not its MU-solution. Indeed, condition (i) does not hold since there is another solution on $\mathbb{R}$, namely $|x|^3$, which coincides with $u(x)$ on $(0, +\infty)$.

2) The function $u(x) = x^3$ defined on $(0; +\infty)$ is a MU-solution to equation (1.3). Indeed, condition (i) holds since the right-hand side of (1.3) is locally Lipschitz continuous in the domain $\{(u_0, u_1) : u_0 > 0\}$ containing the set $\{(u(x), u'(x)) : x > 0\}$. The second sub-condition of (ii) holds since x^3 and $|x|^3$ are two solutions defined on $\mathbb{R} \supset (0; +\infty)$, coinciding with u on $(0; +\infty)$, and having different values at the points $x < 0$ arbitrarily close to 0. Thus, u is non-uniquely extensible outside of $(0, +\infty)$.

3) The function $u(x) = x^3$ defined on $(a; +\infty)$ with $a > 0$ is not a MU-solution to equation (1.3) since condition (ii) does not hold. Indeed, the first sub-condition of (ii) does not hold since u is extensible outside of $(a; +\infty)$. The second sub-condition does not hold since all extensions of u coincide at any point $x > 0$ due to the Lipschitz continuity.

4) The zero function $z(x) \equiv 0$ defined on any interval $I \subset \mathbb{R}$ is a trivial solution to (1.3) but not its MU-solution. Indeed, condition (i) does not hold since for any $a \in I$ the function u_a defined on I by $u_a(x) = \min\{0, (x-a)^3\}$ is another solution to (1.3) coinciding with $z(x)$ on $I \cap (a, +\infty)$. The trivial solution may be considered as a "minimally unique" one.

In this article all MU-solutions to equations (1.1) and (1.2) are classified according to their behavior near the boundaries of their domains. All maximally extended solution can be classified through investigation of possible ways to join several MU-solutions.

2 Preliminary Results. Existence, uniqueness, and continuous dependence of solutions on initial conditions

Consider the equation

$$y^{(n)} + p(x, y, y', \dots, y^{(n-1)})\,|y|^k \operatorname{sgn} y = 0, \quad n \ge 2, \quad k \in \mathbb{R}, \quad 0 < k < 1.$$

Note that the conditions of the classical theorem on the uniqueness of solutions to the Cauchy problem do not hold with such k. Nevertheless, the following assertion holds (see [5], **7.3**; [7]).

Theorem 1. *Let the function $p(x, y_0, \dots, y_{n-1})$ be continuous in x and Lipschitz continuous in $y_0, \dots, y_{n-1}$. Then for any tuple of numbers $x_0, y_0^0, \dots, y_{n-1}^0$ with at least one y_i^0 not equal to zero, the corresponding Cauchy problem $y(x_0) = y_0^0, \dots, y^{(n-1)}(x_0) = y_{n-1}^0$ has a unique solution.*

While the uniqueness conditions hold, the property of continuous dependence of solution on initial data fulfils. (See [9], **V**, Theorem 2.1.)

3 Main Results. Asymptotic classification of solutions to equations (1.1) and (1.2)

In this section an asymptotic classification of all solutions to equation (1.1) is presented, the proof given in section 4. A result concerning asymptotic classification of all solutions to equation (1.2) is presented without proof.

Previously obtained author's results on the asymptotic behavior of solutions to equations (1.1) and (1.2) with $k > 1$ are contained in [5], [6].

Definition 2. A function is called oscillatory near a point $b \in [-\infty, +\infty]$ (or, synonymously, as $x \to b$) if it takes both positive and negative values in any neighborhood of b. A function is called oscillatory (with no point specified) if it is oscillatory near the both boundaries of its domain.

Theorem 2. *Suppose $0 < k < 1$ and $p_0 > 0$. Then all MU-solutions to equation (1.1) are divided into the following three types according to their asymptotic behavior (see Fig. 1). There exist solutions of all these types with arbitrary constants b if mentioned.*

1. Oscillatory solutions defined on $(-\infty, b)$. The distance between their neighboring zeros infinitely increases near $-\infty$ and tends to zero near b. All their derivatives $y^{(j)}$ with $j = 0, \dots, 3$ satisfy the relations $\lim\limits_{x\to b} y^{(j)}(x) = 0$, $\overline{\lim\limits_{x\to-\infty}} \left|y^{(j)}(x)\right| = \infty$. At the points of local extremum the following estimates hold:

$$C_1\,|x-b|^{-\frac{4}{k-1}} \le |y(x)| \le C_2\,|x-b|^{-\frac{4}{k-1}} \tag{3.1}$$

with positive constants C_1 and C_2 depending only on k and p_0.

2. Oscillatory solutions defined on $(b, +\infty)$. The distance between their neighboring zeros tends to zero near b and infinitely increases near $+\infty$. All their derivatives $y^{(j)}$ with $j = 0, \dots, 3$ satisfy the relations $\lim\limits_{x\to b} y^{(j)}(x) = 0$,

$\overline{\lim\limits_{x\to+\infty}} \left|y^{(j)}(x)\right| = \infty$. *At the points of local extremum estimates (3.1) hold with positive constants C_1 and C_2 depending only on k and p_0.*

3. Oscillatory solutions defined on $(-\infty, +\infty)$. All their derivatives $y^{(j)}$ with $j = 0, \dots, 4$ satisfy

$$\overline{\lim_{x\to-\infty}} \left|y^{(j)}(x)\right| = \overline{\lim_{x\to+\infty}} \left|y^{(j)}(x)\right| = \infty.$$

At the points of local extremum the estimates

$$C_1 \, |x|^{-\frac{4}{k-1}} \le |y(x)| \le C_2 \, |x|^{-\frac{4}{k-1}}$$

hold near $-\infty$ and $+\infty$ with positive constants C_1 and C_2 depending only on k and p_0.

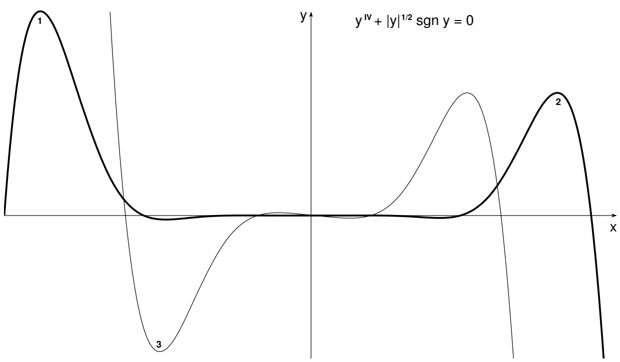

Figure 1. MU-solutions to equation (1.1)

Theorem 3. *Suppose $0 < k < 1$ and $p_0 > 0$. Then all MU-solutions to equation (1.2) are divided into the following thirteen types according to their asymptotic behavior (see Fig. 2). There exist solutions of all these types with arbitrary constants b if mentioned.*

1–2. Defined on semi-axes $(-\infty, b)$ solutions with the power asymptotic behavior near the boundaries of the domain (with the same signs $\pm$):

$$y(x) \sim \pm C_{4k} \, |x|^{-\frac{4}{k-1}}, \qquad x \to -\infty,$$
$$y(x) \sim \pm C_{4k} \, (b-x)^{-\frac{4}{k-1}}, \qquad x \to b-0,$$

where

$$C_{4k} = \left(\frac{4(k+3)(2k+2)(3k+1)}{p_0 \, (k-1)^4} \right)^{\frac{1}{k-1}}.$$

3–4. Defined on $(b, +\infty)$ solutions with the power asymptotic behavior near the boundaries of the domain (with the same signs $\pm$):

$$y(x) \sim \pm C_{4k} \, (x-b)^{-\frac{4}{k-1}}, \qquad x \to b+0,$$
$$y(x) \sim \pm C_{4k} \, x^{-\frac{4}{k-1}}, \qquad x \to +\infty.$$

5. Defined on the whole axis periodic oscillatory solutions. All of them can be received from one solution, say $z(x)$, by the relation

$$y(x) = \lambda^4 z(\lambda^{k-1}x + x_0)$$

with arbitrary $\lambda > 0$ and x_0. So, there exists such a solution with any maximum $h > 0$ and with any period $T > 0$, but not with any pair (h, T).

6–9. Defined on $(-\infty, +\infty)$ solutions having the power asymptotic behavior near $-\infty$ and $+\infty$ (with all sign combinations admitted):

$$y(x) \sim \pm C_{4k}\ |x|^{-\frac{4}{k-1}}, \qquad x \to \pm\infty.$$

10–11. Defined on $(-\infty, +\infty)$ solutions which are oscillatory as $x \to -\infty$ and have the power asymptotic behavior near $+\infty$:

$$y(x) \sim \pm C_{4k}\ x^{-\frac{4}{k-1}}, \qquad x \to +\infty.$$

Each solution has a finite limit of the absolute values of its local extrema as $x \to -\infty$.

12–13. Defined on $(-\infty, +\infty)$ solutions which are oscillatory as $x \to +\infty$ and have the power asymptotic behavior near $-\infty$:

$$y(x) \sim \pm C_{4k}\ |x|^{-\frac{4}{k-1}}, \qquad x \to -\infty.$$

Each solution has a finite limit of the absolute values of its local extrema as $x \to +\infty$.

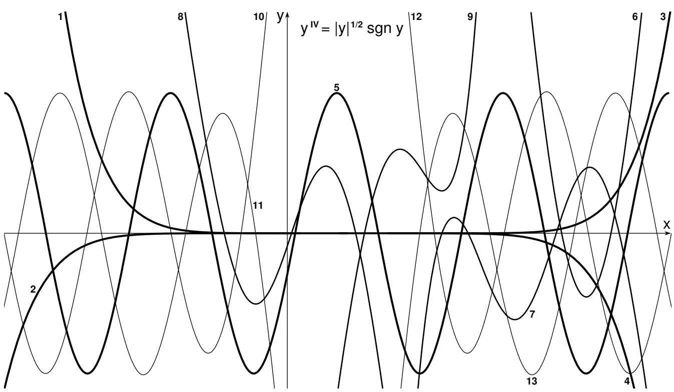

Figure 2. MU-solutions to equation (1.2)

4 Proof of The Main Results

4.1 Phase Sphere

Note that if a function $y(x)$ is a solution to equation (1.1), the same is true for the function

$$z(x) = Ay\,(Bx + C), \tag{4.1}$$

where $A \neq 0$, $B > 0$, and C are any constants satisfying

$$|A|^{k-1} = B^4. \tag{4.2}$$

Indeed, we have

$$\begin{aligned} &z^{(\mathrm{IV})}(x) + p_0 \, |z(x)|^k \operatorname{sgn} z(x) \\ &= AB^4 y^{(\mathrm{IV})}(Bx+C) + p_0 \, |Ay(Bx+C)|^k \operatorname{sgn}(Ay(Bx+C)) \\ &= Ay^{(\mathrm{IV})}(Bx+C)\big(B^4 - |A|^{k-1}\big) = 0. \end{aligned}$$

Any non-trivial solution $y(x)$ to equation (1.1) generates in $\mathbb{R}^4 \setminus \{0\}$ a curve $(y(x), y'(x), y''(x), y'''(x))$. Let us introduce in $\mathbb{R}^4 \setminus \{0\}$ an equivalence relation such that two solutions connected by (4.1)–(4.2) generate equivalent curves, i.e. the curves passing through equivalent points (may be for different x).

We assume that points (y_0, y_1, y_2, y_3) and (z_0, z_1, z_2, z_3) in $\mathbb{R}^4 \setminus \{0\}$ are equivalent if there exists a positive constant λ such that

$$z_j = \lambda^{4+j(k-1)} y_j, \qquad j = 0, 1, 2, 3.$$

The factor space obtained is homeomorphic to the three-dimensional sphere

$$S^3 = \left\{ y \in \mathbb{R}^4 : \quad y_0^2 + y_1^2 + y_2^2 + y_3^2 = 1 \right\}.$$

On this sphere there is exactly one representative of each equivalence class because for any point $(y_0, y_1, y_2, y_3) \in \mathbb{R}^4 \setminus \{0\}$ the equation

$$\lambda^8 y_0^2 + \lambda^{2k+6} y_1^2 + \lambda^{4k+4} y_2^2 + \lambda^{6k+2} y_3^2 = 1$$

has exactly one positive root λ.

It is possible to construct another hyper-surface in $\mathbb{R}^4$ with a single representative of each equivalence class, namely,

$$E = \Big\{ y \in \mathbb{R}^4 : \quad \sum_{j=0}^{3} |y_j|^{\frac{1}{j(k-1)+4}} = 1 \Big\}. \tag{4.3}$$

We define $\varPhi_S : \mathbb{R}^4 \setminus \{0\} \to S^3$ and $\varPhi_E : \mathbb{R}^4 \setminus \{0\} \to E$ as mappings taking each point in $\mathbb{R}^4 \setminus \{0\}$ to the equivalent point in S^3 or E. Note that the restrictions $\varPhi_S\Big|_E$ and $\varPhi_E\Big|_{S^3}$ are inverse homeomorphisms.

Lemma 1. *There is a dynamical system on the sphere S^3 such that all its trajectories can be obtained by the mapping $\varPhi_S$ from the curves generated in $\mathbb{R}^4 \setminus \{0\}$ by nontrivial solutions to equation* (1.1). *Conversely, any nontrivial solution to equation* (1.1) *generates in $\mathbb{R}^4 \setminus \{0\}$ a curve whose image under $\varPhi_S$ is a trajectory of the above dynamical system.*

Proof. First we define on the sphere S^3 a smooth structure using an atlas consisting of eight charts.

The two semi-spheres defined by the inequalities $y_0 > 0$ and $y_0 < 0$ are covered by the charts with the coordinate functions (respectively u_1^+, u_2^+, u_3^+ and u_1^-, u_2^-, u_3^-) defined by the formulae

$$u_j^\pm = y_j\,|y_0|^{-\frac{4+j(k-1)}{4}}\operatorname{sgn} y_0, \qquad j = 1, 2, 3.$$

The semi-spheres defined by the inequalities $y_1 > 0$ and $y_1 < 0$ are covered by the charts with the coordinate functions (respectively v_0^+, v_2^+, v_3^+ and v_0^-, v_2^-, v_3^-) defined as

$$v_j^\pm = y_j\,|y_1|^{-\frac{4+j(k-1)}{k+3}}\operatorname{sgn} y_1, \qquad j = 0, 2, 3.$$

The semi-spheres defined by the inequalities $y_2 > 0$ and $y_2 < 0$ are covered by the charts with the coordinate functions (respectively w_0^+, w_1^+, w_3^+ and w_0^-, w_1^-, w_3^-) defined as

$$w_j^\pm = y_j\,|y_2|^{-\frac{4+j(k-1)}{2k+2}}\operatorname{sgn} y_2, \qquad j = 0, 1, 3.$$

Finally, the semi-spheres defined by the inequalities $y_3 > 0$ and $y_3 < 0$ are covered by the charts with the coordinate functions (respectively g_0^+, g_1^+, g_2^+ and g_0^-, g_1^-, g_2^-) defined as

$$g_j^\pm = y_j\,|y_3|^{-\frac{4+j(k-1)}{3k+1}}\operatorname{sgn} y_3, \qquad j = 0, 1, 2.$$

Note that each of these coordinate functions can be defined by its own formula on the whole corresponding semi-space ($y_j \gtrless 0$) and it takes equivalent points to the same value. This fact facilitates description of the trajectories generated on S^3 by solutions to equation (1.1). To be more precise, by their restrictions on the intervals where some derivative has constant sign.

E. g., when a solution is positive, the trajectory generated can be described by the following differential equations:

$$\begin{aligned}
\frac{du_1^+}{dx} &= y''\,|y|^{-\frac{k+3}{4}}\operatorname{sgn} y - \tfrac{k+3}{4}\,y'^2\,|y|^{-\frac{k+7}{4}} \\
&= |y|^{\frac{k-1}{4}}\left(u_2^+ - \tfrac{k+3}{4}\,{u_1^+}^2\right), \\
\frac{du_2^+}{dx} &= y'''\,|y|^{-\frac{2k+2}{4}}\operatorname{sgn} y - \tfrac{2k+2}{4}\,y'y''\,|y|^{-\frac{2k+6}{4}} \\
&= |y|^{\frac{k-1}{4}}\left(u_3^+ - \tfrac{2k+2}{4}\,u_1^+u_2^+\right), \\
\frac{du_3^+}{dx} &= -p_0\,|y|^{k-\frac{3k+1}{4}} - \tfrac{3k+1}{4}\,y'y'''\,|y|^{-\frac{3k+5}{4}} \\
&= |y|^{\frac{k-1}{4}}\left(-p_0 - \tfrac{3k+1}{4}\,u_1^+u_3^+\right).
\end{aligned}$$

Parameterizing the trajectory by $t_u = \int\limits_{x_0}^{x} |y|^{\frac{k-1}{4}}\,dx$, we obtain its internal description in terms of u_j^+ :

$$\begin{cases} \dfrac{du_1^+}{dt_u} &= u_2^+ - \frac{k+3}{4}\, {u_1^+}^2, \\ \dfrac{du_2^+}{dt_u} &= u_3^+ - \frac{2k+2}{4}\, u_1^+ u_2^+, \\ \dfrac{du_3^+}{dt_u} &= -p_0 - \frac{3k+1}{4}\, u_1^+ u_3^+. \end{cases}$$

The same equations appear for (u_1^-, u_2^-, u_3^-). Similar calculations yield equations for other charts:

$$\begin{cases} \dfrac{dv_0^\pm}{dt_v} &= 1 - \frac{4}{k+3}\, v_0^\pm v_2^\pm, \\ \dfrac{dv_2^\pm}{dt_v} &= v_3^\pm - \frac{2k+2}{k+3}\, {v_2^\pm}^2, \\ \dfrac{dv_3^\pm}{dt_v} &= -p_0 \left|v_0^\pm\right|^k \operatorname{sgn} v_0^\pm - \frac{3k+1}{k+3}\, v_2^\pm v_3^\pm, \end{cases}$$

$$\begin{cases} \dfrac{dw_0^\pm}{dt_w} &= w_1^\pm - \frac{4}{2k+2}\, w_0^\pm w_3^\pm, \\ \dfrac{dw_1^\pm}{dt_w} &= 1 - \frac{k+3}{2k+2}\, w_1^\pm w_3^\pm, \\ \dfrac{dw_3^\pm}{dt_w} &= -p_0 \left|w_0^\pm\right|^k \operatorname{sgn} w_0^\pm - \frac{3k+1}{2k+2}\, {w_3^\pm}^2, \end{cases}$$

$$\begin{cases} \dfrac{dg_0^\pm}{dt_q} &= g_1^\pm + \frac{4}{3k+1}\, p_0 \left|g_0^\pm\right|^{k+1}, \\ \dfrac{dg_1^\pm}{dt_q} &= g_2^\pm + \frac{k+3}{3k+1}\, p_0\, g_1^\pm \left|g_0^\pm\right|^k \operatorname{sgn} g_0^\pm, \\ \dfrac{dg_2^\pm}{dt_q} &= 1 + \frac{2k+2}{3k+1}\, p_0\, g_2^\pm \left|g_0^\pm\right|^k \operatorname{sgn} g_0^\pm. \end{cases}$$

Using a partition of unity one can obtain a dynamical system on the whole sphere S^3 to describe all trajectories generated by nontrivial solutions to equation (1.1). □

4.2 Typical and Non-Typical solutions

Now we consider the space $\mathbb{R}^4$ as the union of its $16 = 2^4$ closed subsets defined according to different combinations of signs of the four coordinates. Denote these sets by $\begin{bmatrix} \pm \\ \pm \\ \pm \\ \pm \end{bmatrix} \subset \mathbb{R}^4$, where each sign $\pm$ can be substituted by $+$, or $-$, or 0 (for boundary points). For example,

$$\begin{bmatrix} + \\ + \\ 0 \\ - \end{bmatrix} = \left\{ y \in \mathbb{R}^4 : y_0 \ge 0, y_1 \ge 0, y_2 = 0, y_3 \le 0, \right\}.$$

Besides, let Ω_- and Ω_+ denote respectively

$$\begin{bmatrix} + \\ - \\ + \\ - \end{bmatrix} \cup \begin{bmatrix} + \\ - \\ + \\ + \end{bmatrix} \cup \begin{bmatrix} + \\ - \\ - \\ + \end{bmatrix} \cup \begin{bmatrix} + \\ + \\ - \\ + \end{bmatrix} \cup \begin{bmatrix} - \\ + \\ - \\ + \end{bmatrix} \cup \begin{bmatrix} - \\ + \\ - \\ - \end{bmatrix} \cup \begin{bmatrix} - \\ + \\ + \\ - \end{bmatrix} \cup \begin{bmatrix} - \\ - \\ + \\ - \end{bmatrix}$$

and

$$\begin{bmatrix}+\\+\\+\\+\end{bmatrix}\cup\begin{bmatrix}+\\+\\+\\-\end{bmatrix}\cup\begin{bmatrix}+\\+\\-\\-\end{bmatrix}\cup\begin{bmatrix}+\\-\\-\\-\end{bmatrix}\cup\begin{bmatrix}-\\-\\-\\-\end{bmatrix}\cup\begin{bmatrix}-\\-\\-\\+\end{bmatrix}\cup\begin{bmatrix}-\\-\\+\\+\end{bmatrix}\cup\begin{bmatrix}-\\+\\+\\+\end{bmatrix}.$$

Note, that the sets Ω_- and Ω_+ cover the whole space $\mathbb{R}^4$, intersect only along their common boundary, and can be obtained from each other using the mapping

$$(y_0, y_1, y_2, y_3) \in \mathbb{R}^4 \mapsto (y_0, -y_1, y_2, -y_3) \in \mathbb{R}^4,$$

which corresponds to changing the sign of the independent variable ($x \mapsto -x$).

Lemma 2. *The sets $\Omega_- \cap S^3$, $\Omega_+ \cap S^3$, $\Omega_- \cap E$, and $\Omega_+ \cap E$ are homeomorphic to the solid torus.*

Proof. It is sufficient to consider $\Omega_+ \cap S^3$. The set Ω_+ is the union of its two homeomorphic subsets

$$\Omega_{++} = \begin{bmatrix}+\\+\\+\\+\end{bmatrix}\cup\begin{bmatrix}+\\+\\+\\-\end{bmatrix}\cup\begin{bmatrix}+\\+\\-\\-\end{bmatrix}\cup\begin{bmatrix}+\\-\\-\\-\end{bmatrix}$$

$$\Omega_{+-} = \begin{bmatrix}-\\-\\-\\-\end{bmatrix}\cup\begin{bmatrix}-\\-\\-\\+\end{bmatrix}\cup\begin{bmatrix}-\\-\\+\\+\end{bmatrix}\cup\begin{bmatrix}-\\+\\+\\+\end{bmatrix}.$$

In order to describe the set $\Omega_{++} \cap S^3$, we use the stereographic projection mapping $S^3 \setminus \{(-1, 0, 0, 0)\}$ onto $\mathbb{R}^3$ (See Figure 3).

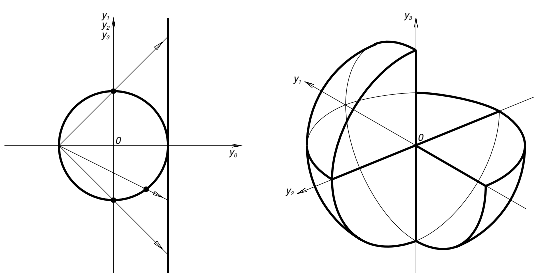

Figure 3. Stereographic projection and its image of $\Omega_{++} \cap S^3$

The image of $\Omega_{++} \cap S^3$ under this projection is contained in the ball of radius 2 and is equal to the union of its two quarters, which is homeomorphic to the 3-dimensional ball. The same is true for $\Omega_{+-} \cap S^3$.

The intersection $\left(\Omega_{++} \cap S^3\right) \cap \left(\Omega_{+-} \cap S^3\right) = \left(\begin{bmatrix}0\\+\\+\\+\end{bmatrix}\cup\begin{bmatrix}0\\-\\-\\-\end{bmatrix}\right) \cap S^3$ maps to the disjoint union of two spherical triangles (2-dimensional figures, not their boundaries). Thus, the set $\Omega_+ \cap S^3$ is homeomorphic to the pair of two balls glued along two disjoint triangles, which is equivalent to the solid torus. □

Lemma 3. *Any trajectory in $\mathbb{R}^4$ generated by a non-trivial solution to* (1.1) *either completely lies inside one of the sets Ω_- and Ω_+ (i. e., in their interior), or consists of two parts, first inside Ω_- and another inside Ω_+ with a single point in their common boundary.*

Proof. For the trajectories generated by solutions to equation (1.1), consider all possible passages between the sets $\begin{bmatrix}\pm\\\pm\\\pm\\\pm\end{bmatrix}$.

Inside Ω_+ the only possible passages are

$$\begin{array}{ccccccc}
\begin{bmatrix}+\\+\\+\\+\end{bmatrix} & \to & \begin{bmatrix}+\\+\\+\\-\end{bmatrix} & \to & \begin{bmatrix}+\\+\\-\\-\end{bmatrix} & \to & \begin{bmatrix}+\\-\\-\\-\end{bmatrix} \\
\uparrow & & & & & & \downarrow \\
\begin{bmatrix}-\\+\\+\\+\end{bmatrix} & \leftarrow & \begin{bmatrix}-\\-\\+\\+\end{bmatrix} & \leftarrow & \begin{bmatrix}-\\-\\-\\+\end{bmatrix} & \leftarrow & \begin{bmatrix}-\\-\\-\\-\end{bmatrix}
\end{array}, \tag{4.4}$$

inside Ω_- they are

$$\begin{array}{ccccccc}
\begin{bmatrix}+\\-\\+\\-\end{bmatrix} & \leftarrow & \begin{bmatrix}+\\-\\+\\+\end{bmatrix} & \leftarrow & \begin{bmatrix}+\\-\\-\\+\end{bmatrix} & \leftarrow & \begin{bmatrix}+\\+\\-\\+\end{bmatrix} \\
\downarrow & & & & & & \uparrow \\
\begin{bmatrix}-\\-\\+\\-\end{bmatrix} & \to & \begin{bmatrix}-\\+\\+\\-\end{bmatrix} & \to & \begin{bmatrix}-\\+\\-\\-\end{bmatrix} & \to & \begin{bmatrix}-\\+\\-\\+\end{bmatrix}
\end{array}, \tag{4.5}$$

and the only possible passages between Ω_- and Ω_+ are

$$\begin{array}{ll}
\begin{bmatrix}+\\+\\+\\-\end{bmatrix} \leftarrow \begin{bmatrix}+\\-\\+\\-\end{bmatrix} \to \begin{bmatrix}+\\-\\-\\-\end{bmatrix}, & \begin{bmatrix}-\\-\\+\\+\end{bmatrix} \leftarrow \begin{bmatrix}+\\-\\+\\+\end{bmatrix} \to \begin{bmatrix}+\\+\\+\\+\end{bmatrix}, \\
\begin{bmatrix}-\\-\\-\\+\end{bmatrix} \leftarrow \begin{bmatrix}+\\-\\-\\+\end{bmatrix} \to \begin{bmatrix}+\\-\\-\\-\end{bmatrix}, & \begin{bmatrix}+\\+\\+\\+\end{bmatrix} \leftarrow \begin{bmatrix}+\\+\\-\\+\end{bmatrix} \to \begin{bmatrix}+\\+\\-\\-\end{bmatrix}, \\
\begin{bmatrix}-\\-\\-\\+\end{bmatrix} \leftarrow \begin{bmatrix}-\\+\\-\\+\end{bmatrix} \to \begin{bmatrix}-\\+\\+\\+\end{bmatrix}, & \begin{bmatrix}+\\+\\-\\-\end{bmatrix} \leftarrow \begin{bmatrix}-\\+\\-\\-\end{bmatrix} \to \begin{bmatrix}-\\-\\-\\-\end{bmatrix}, \\
\begin{bmatrix}+\\+\\+\\-\end{bmatrix} \leftarrow \begin{bmatrix}-\\+\\+\\-\end{bmatrix} \to \begin{bmatrix}-\\+\\+\\+\end{bmatrix}, & \begin{bmatrix}-\\-\\-\\-\end{bmatrix} \leftarrow \begin{bmatrix}-\\-\\+\\-\end{bmatrix} \to \begin{bmatrix}-\\-\\+\\+\end{bmatrix},
\end{array}$$

always from Ω_- to Ω_+.

So, any trajectory generated by a non-trivial solution can perform only one passage between Ω_- and Ω_+, which can be only from Ω_- to Ω_+. $\square$

Lemma 4. *There exist trajectories of all three types mentioned in Lemma 3, namely: (i) trajectories lying completely in Ω_-, (ii) trajectories lying completely in Ω_+, (iii) trajectories with a single passage $\Omega_- \to \Omega_+$.*

Proof. Any solution to (1.1) with initial data corresponding to a point from $\Omega_- \cap \Omega_+$ generates a trajectory of the 3rd type. E. g., the solution with initial data $y'(0) = 0$, $y(0) = y''(0) = y'''(0) = 1$ generates a trajectory with the passage

$$\begin{bmatrix}+\\-\\+\\+\end{bmatrix} \subset \Omega_- \to \begin{bmatrix}+\\+\\+\\+\end{bmatrix} \subset \Omega_+ .$$

If there exists a solution $y(x)$ to (1.1) generating a trajectory lying completely in Ω_-, then the function $z(x) = y(-x)$ is also a solution to (1.1) and generates a trajectory completely lying in Ω_+. So, we have to prove existence of a trajectory of the first type.

Assume the converse. Then any trajectory passing through a point $s \in \Omega_- \cap S^3$ must reach the boundary $\partial\Omega_- \cap S^3$. Thus we obtain the mapping $\Omega_- \cap S^3 \to \partial\Omega_- \cap S^3$. To prove its continuity we represent it as

$$s \in \Omega_- \cap S^3 \mapsto \mathbf{Traj_{p_0}}(s,\ \xi(s)) \in \partial\Omega_- \cap S^3.$$

Here $\mathbf{Traj_{p_0}}(s,t)$ is the point in S^3 reached at the time t by the trajectory of the dynamical system on the sphere that passed s at the time 0. The mapping $\mathbf{Traj_{p_0}} : S^3 \times \mathbb{R} \to S^3$ is continuous according to the general properties of differential equations.

The function $\xi : \Omega_- \cap S^3 \to \mathbb{R}$ gives the time t at which the trajectory passing through the given point of Ω_- at $t_0 = 0$ reaches $\partial\Omega_-$. Now we prove continuity of ξ.

Suppose $\xi(s_1) = t_1$ and $\varepsilon > 0$. Then, since $\mathbf{Traj_{p_0}}(s_1, t_1 + \varepsilon)$ is inside Ω_+, there exists a neighborhood U_+ of s_1 such that for any $s \in U_+$ the point $\mathbf{Traj_{p_0}}(s, t_1 + \varepsilon)$ is also inside Ω_+. So, we have $\xi(s) < t_1 + \varepsilon$ for all $s \in U_+$.

Similarly, since $\mathbf{Traj_{p_0}}(s_1, t_1 - \varepsilon)$ is inside Ω_-, there exists a neighborhood U_- of s_1 such that for any $s \in U_-$ the point $\mathbf{Traj_{p_0}}(s, t_1 - \varepsilon)$ is also inside Ω_-, whence $\xi(s) > t_1 - \varepsilon$.

So, for all $s \in U_- \cap U_+$ we have $|\xi(s) - t_1| < \varepsilon$. Thus $\xi(s)$ is continuous on $\Omega_- \cap S^3$ and we have the continuous mapping $\Omega_- \cap S^3 \to \partial\Omega_- \cap S^3$ whose restriction to $\partial\Omega_- \cap S^3$ is the identity map. In other words, we have the composition

$$\partial\Omega_- \cap S^3 \hookrightarrow \Omega_- \cap S^3 \to \partial\Omega_- \cap S^3,$$

which is the identity map, inducing the identity map on the homology groups:

$$H_2(\partial\Omega_- \cap S^3) \to H_2(\Omega_- \cap S^3) \to H_2(\partial\Omega_- \cap S^3).$$

Since $\Omega_- \cap S^3$ and $\partial\Omega_- \cap S^3$ are homeomorphic to the solid torus and the torus surface respectively, the above composition can be written as $\mathbb{Z} \to 0 \to \mathbb{Z}$, which cannot be the identity mapping. This contradiction proves the lemma. □

Lemma 5. *Suppose $y(x)$ is a MU-solution to equation* (1.1). *Then neither $y(x)$ nor any of its derivatives $y'(x)$, $y''(x)$, $y'''(x)$ can have constant sign near any boundary of their domain.*

Proof. We prove it for $y(x)$. For the derivatives the proof is just similar. We'll consider the right boundary. For the left boundary the proof is the same because if $y(x)$ is a solution of equation (1.1) then $x \mapsto y(-x)$ is also its solution. Suppose $y(x)$ is defined on an interval (x_-, x_+), bounded or not, and is positive in a neighborhood of x_+. Then $y'''(x)$, due to (1.1), is monotonically decreasing to a finite or infinite limit as $x \to x_+$. Then $y'''(x)$ ultimately has a constant sign. In the same way, $y''(x)$, $y'(x)$, and $y(x)$ itself are all ultimately monotone and have finite or infinite limits as $x \to x_+$.

If all these limits are zero, then $y(x)$, which is ultimately positive, is decreasing to 0. Hence, $y'(x)$ is ultimately negative and increasing to 0. Similarly, $y''(x)$ is ultimately positive and decreasing to 0, $y'''(x)$ is ultimately negative and increasing to 0. This contradicts to equation (1.1), since $y(x)$ is ultimately positive, whence $y^{(\mathrm{IV})}(x)$ is ultimately negative and $y'''(x)$ is decreasing. So, at least one of the limits mentioned is non-zero.

Suppose $x_+ = +\infty$. Then all limits must be infinite and have the same sign, which contradicts to equation (1.1).

Now suppose $x_+ < +\infty$. If all limits mentioned are infinite, they must have the same sign, which contradicts to equation (1.1). If either one of the limits is finite, then all other limits are finite, too. This is impossible for a MU-solution since at least one of the limits is non-zero.

These contradictions prove the lemma. $\square$

Thus, no trajectory generated in $\mathbb{R}^4$ by a non-trivial solution to (1.1) can ultimately rest in one of the sets $\begin{bmatrix} \pm \\ \pm \\ \pm \\ \pm \end{bmatrix}$.

Corollary 1. All MU-solutions to equation (1.1), *as well as their derivatives, are oscillatory near both boundaries of their domains.*

Note that according to Lemma 3 we can distinguish two types of asymptotic behavior of oscillatory MU-solutions to equation (1.1), near the right boundaries of their domains.

Definition 3. An oscillatory MU-solution to equation (1.1) is called *typical* (to the right) if ultimately this solution and its derivatives change their signs according to scheme (4.4), and *non-typical* if according to (4.5).

4.3 Asymptotic behavior of typical solutions

This section is devoted to the asymptotic behavior of typical (to the right) solutions to equation (1.1), i. e. those generating trajectories ultimately lying inside Ω_+.

Since such a trajectory ultimately admits only the passages shown in (4.4), there exists an increasing sequence of the points $x_0''' < x_0'' < x_0' < x_0 < x_1''' < x_1'' < x_1' < x_1 < \dots$ such that $y(x_j) = y'(x_j') = y''(x_j'') = y'''(x_j''') = 0$ $(j = 0, 1, 2, \dots)$, and each point is a zero only for one of the functions $y(x)$, $y'(x)$, $y''(x)$, $y'''(x)$ (see Fig. 4). The points x_j, x_j', x_j'', x_j''' will be called the *nodes* of the solution $y(x)$.

For solutions generating trajectories completely lying inside Ω_+, the sequences of their nodes can be indexed by all integers (negative ones, too).

Lemma 6. *Any typical solution $y(x)$ to equation* (1.1) *satisfies at its nodes the following inequalities:*

$$\left|y(x_j')\right| < \left|y(x_{j+1}''')\right| < \left|y(x_{j+1}'')\right| < \left|y(x_{j+1}')\right|, \tag{4.6}$$

$$\left|y'(x_j'')\right| < \left|y'(x_j)\right| < \left|y'(x_{j+1}''')\right| < \left|y'(x_{j+1}'')\right|, \tag{4.7}$$

$$\left|y''(x_j''')\right| < \left|y''(x_j')\right| < \left|y''(x_j)\right| < \left|y''(x_{j+1}''')\right|, \tag{4.8}$$

$$\left|y'''(x_j)\right| < \left|y'''(x_{j+1}'')\right| < \left|y'''(x_{j+1}')\right| < \left|y'''(x_{j+1})\right|. \tag{4.9}$$

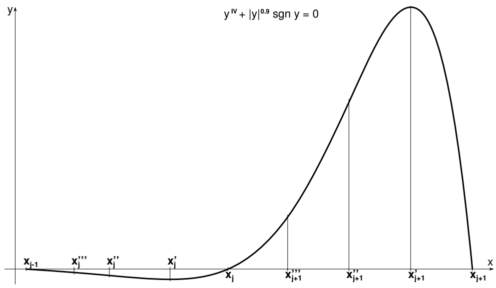

Figure 4. Nodes of a solution

Proof.

$$\frac{p_0}{k+1}\left(\left|y(x_j')\right|^{k+1} - \left|y(x_{j+1}''')\right|^{k+1}\right) = -p_0 \int_{x_j'}^{x_{j+1}'''} y'(x)\,|y(x)|^k \operatorname{sgn} y(x)\,dx$$
$$= \int_{x_j'}^{x_{j+1}'''} y'(x)\,y^{(\mathrm{IV})}(x)\,dx = y'(x)\,y'''(x)\Big|_{x_j'}^{x_{j+1}'''} - \int_{x_j'}^{x_{j+1}'''} y''(x)\,y'''(x)\,dx,$$

which is negative since $y''(x)y'''(x) > 0$ for all $x \in \left[x_j',\, x_{j+1}'''\right)$ and $y'(x_j') = y'''(x_{j+1}''') = 0$. This gives the first of inequalities (4.6), whereas the rest of them follow from $y(x)y'(x) > 0$ on the interval $\left[x_{j+1}''',\, x_{j+1}'\right)$.

Similarly, for the first of inequalities (4.7) we have

$$y'(x_j'')^2 - y'(x_j)^2 = -2\int_{x_j''}^{x_j} y'(x)y''(x)\,dx$$
$$= -2y(x)y''(x)\Big|_{x_j''}^{x_j} + 2\int_{x_j''}^{x_j} y(x)y'''(x)\,dx < 0,$$

since $y(x_j) = y''(x_j'') = 0$ and $y(x)y'''(x) < 0$ on $\left[x_j'',\, x_j\right)$. The rest ones follow from the inequality $y'(x)y''(x) > 0$ on $\left[x_j,\, x_{j+1}''\right)$.

In the same way, for the first of (4.8) we have

$$y''(x_j''')^2 - y''(x_j')^2 = -2\int_{x_j'''}^{x_j'} y''(x)y'''(x)\,dx$$
$$= -2y'(x)y'''(x)\Big|_{x_j'''}^{x_j'} + 2\int_{x_j'''}^{x_j'} y'(x)y^{(\mathrm{IV})}(x)\,dx < 0,$$

since $y'(x)y^{(\mathrm{IV})}(x) = -p_0 y'(x)\,|y|^k \operatorname{sgn} y(x) < 0$ on $\left[x_j''',\, x_j'\right)$ and $y'(x_j') = y'''(x_j''') = 0$. The rest ones follow from $y''(x)y'''(x) > 0$ on $\left[x_j',\, x_{j+1}'''\right)$.

Finally, for the first inequality of (4.9) we have

$$y'''(x_j)^2 - y'''(x''_{j+1})^2 = -2\int_{x_j}^{x''_{j+1}} y'''(x)y^{(\mathrm{IV})}(x)\,dx$$
$$= 2p_0\int_{x_j}^{x''_{j+1}} y'''(x)\,|y(x)|^k \operatorname{sgn} y(x)\,dx = 2p_0\,y''(x)\,|y(x)|^k \operatorname{sgn} y(x)\Big|_{x_j}^{x''_{j+1}}$$
$$- 2kp_0\int_{x_j}^{x''_{j+1}} y''(x)\,y'(x)\,|y(x)|^{k-1}\,dx < 0,$$

since $y'(x)y''(x) > 0$ on $\left[x_j,\, x''_{j+1}\right)$ and $y(x_j) = y''(x''_{j+1}) = 0$, whereas the rest inequalities follow from $y'''(x)y^{(\mathrm{IV})}(x) > 0$ on $\left[x''_{j+1},\, x_{j+1}\right)$.

Note that the function $y(x)$ on $[x_j,\, x''_{j+1}]$ vanishes only at the point x_j and $y'(x_j) \neq 0$. Hence, $\int_{x_j}^{x''_{j+1}} y''(x)\,y'(x)\,|y(x)|^{k-1}\,dx$ with $k-1 > -1$ converges. $\square$

So, the absolute values of the local extrema of any typical solution to equation (1.1) form a strictly increasing sequence. The same holds for its first, second, and third derivatives.

Hereafter we need some extra notations. Put

$$\Omega_+^1 = \mathbf{Traj_1}(\Omega_+ \cap S^3, 1) \subset S^3.$$

This is a compact subset of the interior of Ω_+ containing ultimate parts of all trajectories generated by typical solutions to equation (1.1) with $p_0 = 1$. As for solutions generating the curves in $\mathbb{R}^4$ completely lying in Ω_+, the trajectories related completely lie in Ω_+^1.

Besides, we define the compact sets

$$K_i = \left\{\, a \in \Omega_+^1 :\; a_i = 0 \,\right\}$$

and the functions $\xi_j : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R},\ j = 0,1,2,3$, taking each $a \in \mathbb{R}^4 \setminus \{0\}$ to the minimal positive zero of the derivative $y^{(j)}(x)$ of the solution to the initial data problem

$$\begin{cases} y^{(\mathrm{IV})}(x) + |y(x)|^k \operatorname{sgn} y(x) = 0, \\ y^{(j)}(0) = a_j, \qquad j = 0,1,2,3. \end{cases} \tag{4.10}$$

Further, to each solution $y(x)$ to equation (1.1) we associate the function

$$F_y(x) = \sum_{j=0}^{3} \left|\rho y^{(j)}(x)\right|^{\frac{1}{j(k-1)+4}} \qquad \text{with } \rho = p_0^{\frac{1}{k-1}}. \tag{4.11}$$

The notation F_y does not use p_0, since non-trivial functions cannot be solutions to equation (1.1) with different values of p_0.

Lemma 7. *The restrictions* $\xi_i\big|_{K_j}$, $i,j = 0,1,2,3$, *are continuous.*

Proof. First we prove continuity of ξ_i at $a \in \Omega_+$ with $a_i > 0$. Suppose $\xi_i(a) = x_i$ and $\varepsilon > 0$.

We can assume that ε is sufficiently small to be less than x_i and to provide, for the solution $y(x)$ to (4.10), the inequalities $y^{(i)}(x-\varepsilon) > 0$ on $[0, x_i - \varepsilon]$ and $y^{(i)}(x_i+\varepsilon) < 0$. In this case the point a has a neighborhood $U \subset \Omega_+$ such that the above inequalities are satisfied for all solutions to (4.10) with initial data $a' \in U$. Hence, $|\xi_i(a') - x_i| < \varepsilon$. Continuity of ξ_i at $a \in \Omega_+$ with $a_i > 0$ is proved.

In the same way it is proved at $a \in \Omega_+$ with $a_i < 0$. Since $a_i \neq 0$ if $a \in K_j$, $i \neq j$, we have proved continuity of the restriction $\xi_i\big|_{K_j}$ in the case $i \neq j$. As for $\xi_i\big|_{K_i}$, note that between two zeros of $y^{(i)}(x)$ there exists a zero x_j of another derivative $y^{(j)}(x)$. The values $y^{(m)}(x_j)$, $m = 0,1,2,3$, due to continuity of $\xi_j\big|_{K_i}$, depend continuously on $a \in K_i$, whereas the restriction $\xi_i\big|_{K_j}$ depends continuously on these values. This proves continuity of the restriction $\xi_i\big|_{K_i}$.
□

Lemma 8. *For any $k \in (0;1)$ there exist $Q > q > 1$ such that for any typical solution $y(x)$ to equation* (1.1) *the values of all expressions*

$$\begin{array}{lll}
\left|\dfrac{y(x'''_{j+1})}{y(x'_j)}\right|^{\frac14}, & \left|\dfrac{y(x''_j)}{y(x'''_j)}\right|^{\frac14}, & \left|\dfrac{y(x'_j)}{y(x''_j)}\right|^{\frac14}, \\
\left|\dfrac{y'(x_j)}{y'(x''_j)}\right|^{\frac{1}{k+3}}, & \left|\dfrac{y'(x'''_{j+1})}{y'(x_j)}\right|^{\frac{1}{k+3}}, & \left|\dfrac{y'(x''_j)}{y'(x'''_j)}\right|^{\frac{1}{k+3}}, \\
\left|\dfrac{y''(x'_j)}{y''(x'''_j)}\right|^{\frac{1}{2k+2}}, & \left|\dfrac{y''(x_j)}{y''(x'_j)}\right|^{\frac{1}{2k+2}}, & \left|\dfrac{y''(x'''_{j+1})}{y''(x_j)}\right|^{\frac{1}{2k+2}}, \\
\left|\dfrac{y'''(x''_{j+1})}{y'''(x_j)}\right|^{\frac{1}{3k+1}}, & \left|\dfrac{y'''(x'_j)}{y'''(x_j)}\right|^{\frac{1}{3k+1}}, & \left|\dfrac{y'''(x_j)}{y'''(x'_j)}\right|^{\frac{1}{3k+1}}
\end{array}$$

with sufficiently large j are contained in the segment $[q, Q]$.

Proof. Let us define the continuous functions $\psi_{ijl} : K_i \to \mathbb{R}$ (all indices i, j, l are from 0 to 3 and pairwise different) taking each point $a \in K_i$ to the ratio of the absolute values of the j-th derivative of the solution $y(x)$ to (4.10) at 0 and at the next point where the l-th derivative vanishes, i. e. $\psi_{ijl}(a) = \left|\dfrac{a_j}{y^{(j)}(\xi_l(a))}\right|$ (both the numerator and the denominator are non-zero if $a \in K_i$).

Due to Lemma 6, each function ψ_{ijl} at all points of the compact set K_i is positive and less than 1. Hence $0 < \inf\limits_{K_i} \psi_{ijl}(a) \le \sup\limits_{K_i} \psi_{ijl}(a) < 1$.

Now consider an arbitrary typical solution $y(x)$ to (1.1) and two its nodes, say x'_j and x'''_{j+1}, with sufficiently large numbers such that the related points in S^3 belong to Ω^1_+. In this case we can choose constants $A \neq 0$ and $B > 0$ such that the function $z(x) = Ay(Bx + x'_j)$ is a solution to (4.10) with $a \in K_1$. Indeed, this is equivalent to existence of $A \neq 0$ and $B > 0$ such that

$$\begin{cases} |A|^{k-1} = B^4 p_0, \\ \sum_{m=0,2,3} \left(AB^m y^{(m)}(x'_j)\right)^2 = 1, \end{cases}$$

which follows from existence of a root A to the equation

$$\left(y(x_j')\right)^2 A^2 + \left(y''(x_j')\right)^2 p_0^{-1} \, |A|^{k+1} + \left(y'''(x_j')\right)^2 p_0^{-\frac{3}{2}} \, |A|^{\frac{3k+1}{2}} = 1.$$

The value $\left|y(x_{j+1}''')/y(x_j')\right|^{\frac{1}{4}}$ is equal to this for $z(x)$ at $\xi_3(a)$ and 0, where $a_0 = |A|$, $a_1 = 0$, $a_2 = |A|\, B^2$, $a_3 = |A|\, B^3$, i. e. equal to $\psi_{103}(a)^{-\frac{1}{4}}$. Put $q = \left(\sup\limits_{K_1} \psi_{103}(a)\right)^{-\frac{1}{4}}$, $Q = \left(\inf\limits_{K_1} \psi_{103}(a)\right)^{-\frac{1}{4}}$ and obtain the statement of the lemma for the first ratio. The same procedure can be used for others. Then we just choose the minimum of 12 values of q and the maximum of 12 values of Q. □

Lemma 9. *The domain of any typical (to the right) solution $y(x)$ to equation* (1.1) *is right-unbounded and*

$$\overline{\lim_{x\to+\infty}} \left|y^{(n)}(x)\right| = +\infty, \qquad n = 0, 1, 2, 3.$$

Proof. It follows from Lemma 8 that the absolute values of the neighboring local extrema of any typical solution for sufficiently large number, say for $j \ge J$, satisfy the inequality $\left|y(x_{j+1}')\right| \ge q^{12} \left|y(x_j')\right|$ with some $q > 1$, whence

$$\left|y(x_j')\right| \ge q^{12(j-J)} \left|y(x_J')\right| \qquad \text{and} \qquad \lim_{j\to\infty} \left|y(x_j')\right| = +\infty. \tag{4.12}$$

In particular, for $n = 0$ this yields $\overline{\lim} \left|y^{(n)}(x)\right| = +\infty$ as x tends to the right boundary of the domain. Other n are treated similarly.

In order to prove that the domain of the typical (to the right) solution $y(x)$ is right-unbounded, consider the function

$$Y(x) = \sum_{j=0}^{3} \left|y^{(j)}(x)\right|^{\beta_j} \quad \text{with} \quad \beta_j = \frac{5}{4 - j(1-k)} > 1.$$

It is a positive C^1 function. Its derivative can be estimated by using the inequality $\left|y^{(j)}(x)\right| < Y(x)^{1/\beta_j}$ as follows.

$$\begin{aligned} |Y'(x)| &\le \sum_{j=0}^{2} \beta_j \left|y^{(j)}(x)\right|^{\beta_j - 1} \left|y^{(j+1)}(x)\right| + \beta_3 \left|y'''(x)\right|^{\beta_3 - 1} |y(x)|^k \\ &< \sum_{j=0}^{2} \beta_j Y(x)^{1 - 1/\beta_j + 1/\beta_{j+1}} + \beta_3 Y(x)^{1 - 1/\beta_3 + k/\beta_0}. \end{aligned}$$

Since both $1 - \dfrac{1}{\beta_j} + \dfrac{1}{\beta_{j+1}}$ and $1 - \dfrac{1}{\beta_3} + \dfrac{k}{\beta_0}$ are equal to $\dfrac{4+k}{5}$, we have

$$\left|\frac{d}{dx} Y(x)^{(1-k)/5}\right| = \frac{1-k}{5} Y(x)^{-(4+k)/5} \, |Y'(x)| < \frac{1-k}{5} \sum_{j=0}^{3} \beta_j,$$

whence both $Y(x)^{(1-k)/5}$ and $Y(x)$ are bounded on any bounded interval. Now it follows from (4.12) that the domain is right-unbounded. □

Lemma 10. *For any $k \in (0;1)$ there exist positive constants $m \le M$ such that for any typical solution $y(x)$ to equation* (1.1) *the distance between its neighboring points of local extremum, x'_j and x'_{j+1}, ultimately satisfies the estimates*

$$m \le (x'_{j+1} - x'_j)\, F_y(x'_j)^{k-1} \le M \tag{4.13}$$

with the function $F_y(x)$ defined by (4.11).

Proof. Put $E_+ = \Phi_E\left(\Omega^1_+\right)$. It is a compact subset of the set E defined by (4.3) and lying inside Ω_+. Put

$$m = \inf\{\xi_1(a):\ a \in E_+,\, a_1 = 0\} > 0,$$

$$M = \sup\{\xi_1(a):\ a \in E_+,\, a_1 = 0\} < \infty.$$

Let $y(x)$ be a typical solution to equation (1.1), x'_j and x'_{j+1} be neighboring points of its local extremum. We can choose positive constants A and B such that the function $z(x) = Ay(Bx + x'_j)$ is a solution to equation (1.1) with $p_0 = 1$ and its data at zero correspond to some point in E, i. e. $F_z(0) = 1$. It is sufficient for this to find a positive solution to the system

$$\begin{cases} A^{k-1} = B^4 p_0, \\ \sum\limits_{i=0}^{3} \left|AB^i y^{(i)}(x'_j)\right|^{\frac{1}{i(k-1)+4}} = 1, \end{cases}$$

namely

$$A = \left(\sum_{i=0}^{3}\left|\frac{y^{(i)}(x'_j)}{p_0^{i/4}}\right|^r\right)^{-4}, \quad B = \left(\sum_{i=0}^{3}\left|\rho y^{(i)}(x'_j)\right|^r\right)^{1-k} = F_y(x_j)^{1-k},$$

where $r = 1/(i(k-1)+4)$. Moreover, for local extrema with sufficiently large numbers, the point defined in $\mathbb{R}^4$ by the data of the function $z(x)$ at zero belongs to E_+. Hence the first positive point L of local extremum of $z(x)$ belongs to $[m, M]$, whence the difference $x'_{j+1} - x'_j$ is equal to LB and satisfies (4.13). □

Lemma 11. *For any $k \in (0;1)$ there exists a constant $\theta > 0$ such that for any $p_0 > 0$ the local extrema of any typical solution $y(x)$ to equation* (1.1), *ultimately satisfy the inequality*

$$\left|y(x'_j)\right| \ge \theta p_0^{\frac{1}{1-k}} F_y(x'_j)^4 . \tag{4.14}$$

Proof. Let $y(x)$ be a typical solution to equation (1.1) with $p_0 = 1$ and x'_j be its local extremum point with sufficiently large number.

Put $\theta = \inf\{|a_0|:\ a \in E_+,\, a_1 = 0\} > 0$ and choose a constant $\lambda > 0$ such that the data at zero for the solution $z(x) = \lambda^4 y(\lambda^{k-1}x + x'_j)$ correspond to some point in E_+. Then $|z(0)| \ge \theta$ and $F_z(0) = 1$, whence $|z(0)| \ge \theta F_z(0)^4$. Since $z(0) = \lambda^4 y(x'_j)$ and $F_z(0) = \lambda F_y(x'_j)$, we have also $\left|y(x'_j)\right| \ge \theta F_y(x'_j)^4$. So, the lemma is proved for the case $p_0 = 1$.

If $y(x)$ is a typical solution to equation (1.1) with arbitrary $p_0 > 0$, then the function $Y(x) = p_0^{\frac{1}{k-1}} y(x)$ is a typical solution to equation (1.1) with $p_0 = 1$ and hence ultimately satisfies the inequality $\left|Y(x'_j)\right| \geq \theta F_Y(x'_j)^4$. The functions $F_Y(x)$ and $F_y(x)$ are defined by (4.11) with ρ equal to 1 and $p_0^{\frac{1}{k-1}}$, respectively. So, they equal each other providing inequality (4.14). $\square$

Remark 3. For typical solutions to (1.1) with their corresponding curves lying completely in Ω_+, the statements of Lemmas 8, 10, and 11 hold in the whole domain, not only ultimately.

Theorem 4. *For any real $k \in (0;1)$ and $p_0 > 0$ there exist constants $C_1 > 0$ and $C_2 > 0$ such that for any typical (to the right) solution $y(x)$ to equation (1.1) there exists x' such that all local extrema of $y(x)$ at points $x'_j > x'$ satisfy the inequalities*

$$C_1\,(x'_j - x')^{\frac{4}{1-k}} < \left|y(x'_j)\right| < C_2\,(x'_j - x')^{\frac{4}{1-k}}.$$

Proof. Let x'_J and x'_{J+1} be two neighboring points of local extremum of a solution $y(x)$ such that the statements of Lemmas 8, 10, and 11 hold for all $j \geq J$.

According to these Lemmas, for all $j > l > J$ we have

$$(x'_{l+1} - x'_l)\,F_y(x'_l)^{k-1} \leq M,$$
$$\left|y(x'_j)\right|^{\frac{1}{4}} \geq q^{3(j-l)}\,|y(x'_l)|^{\frac{1}{4}} \geq q^{3(j-l)}\theta^{\frac{1}{4}} F_y(x'_l) p_0^{\frac{1}{4(1-k)}}$$

with some $M > 0$, $q > 1$, $\theta > 0$, whence

$$\begin{aligned}\left|y(x'_j)\right|^{-\frac{1-k}{4}}(x'_{l+1} - x'_l) &\leq q^{-3(1-k)(j-l)}\,\theta^{-\frac{1-k}{4}}\,F_y(x'_l)^{k-1}(x'_{l+1} - x'_l)\\ &\leq q^{-3(1-k)(j-l)}\,\theta^{-\frac{1-k}{4}}\,M.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|y(x'_j)\right|^{-\frac{1-k}{4}}(x'_j - x'_J) &\leq \theta^{-\frac{1-k}{4}}\,M \sum_{l=J}^{j-1} q^{-3(1-k)(j-l)}\\ &= \theta^{-\frac{1-k}{4}}\,M \sum_{s=1}^{j-J} q^{-3(1-k)s} < \frac{\theta^{-\frac{1-k}{4}}\,M\,q^{-3(1-k)}}{1 - q^{-3(1-k)}}.\end{aligned}$$

From the other hand,

$$(x'_{l+1} - x'_l)\,F_y(x'_l)^{k-1} \geq m,$$
$$\left|y(x'_j)\right|^{\frac{1}{4}} \leq Q^{3(j-l)}\,|y(x'_l)|^{\frac{1}{4}} \leq Q^{3(j-l)} p_0^{\frac{1}{4(1-k)}}\,F_y(x'_l)$$

with some $m > 0$, $Q > 1$, whence

$$\begin{aligned}\left|y(x'_j)\right|^{-\frac{1-k}{4}}(x'_{l+1} - x'_l) &\geq Q^{-3(1-k)(j-l)} p_0^{-\frac{1}{4}} F_y(x'_l)^{k-1}(x'_{l+1} - x'_l)\\ &\geq Q^{-3(1-k)(j-l)} p_0^{-\frac{1}{4}}\,m.\end{aligned}$$

Therefore,

$$\left|y(x_j')\right|^{-\frac{1-k}{4}}(x_j'-x_J')\geq p_0^{-\frac{1}{4}}\,m\sum_{l=J}^{j-1}Q^{-3(1-k)(j-l)}\geq p_0^{-\frac{1}{4}}\,m\,Q^{-3(1-k)}$$

and there exist some constants $c_1>0$ and $c_2>0$ such that

$$c_1<\left|y(x_j')\right|^{-\frac{1-k}{4}}(x_j'-x_J')<c_2,$$

whence

$$c_2^{-\frac{4}{1-k}}(x_j'-x_J')^{\frac{4}{1-k}}<\left|y(x_j')\right|<c_1^{-\frac{4}{1-k}}(x_j'-x_J')^{\frac{4}{1-k}},$$

which completes the proof. $\square$

With the help of these results we can describe the asymptotic behavior of nontrivial solutions to equation (1.1) and prove Theorem 2.

First, for solutions to equation (1.1) generating in $\mathbb{R}^4$ curves lying entirely in Ω_+, we describe their asymptotic behavior near the left boundary of the domain.

Lemma 12. *Suppose* $y(x)$ *is a typical to the left solution to equation* (1.1) *with derivatives changing their signs according to scheme (4.4). Then the domain of* $y(x)$ *is left-bounded and the functions* $y(x), y'(x), y''(x), y'''(x)$ *tend to zero as* x *tends to the left boundary.*

Using the substitution $x\mapsto -x$ we can describe the asymptotic behavior of non-typical solutions near the right boundaries of their domains. Combining these results we obtain Theorem 2.

Remark 4. The proof of Theorem 3 is based on the method used to obtain the asymptotic classification of solutions to equation (1.2) for $k>1$ (see [5], Ch. 7), but is more complicated.

Remark 5. The existence of solutions of some types mentioned in Theorem 2 and Theorem 3 is known also for equations (1.1) and (1.2) with the constant p_0 replaced by a function $p(x)$ satisfying some conditions. See for example [10], Ch. IV, §15–16. However, for the constant coefficient it becomes now possible to describe the asymptotic behavior of all MU-solutions, thus obtaining their complete qualitative picture.

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