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Cordial Volterra Integral Equations and Singular Fractional Integro-Differential Equations in Spaces of Analytic Functions*

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Abstract. We study general cordial Volterra integral equations of the second kind and certain singular fractional integro-differential equation in spaces of analytic functions. We characterize properties of the cordial Volterra integral operator in these spaces, including compactness and describe its spectrum. This enables us to obtain conditions under which these equations have a unique analytic solution. We also consider approximate solution of these equations and prove exponential convergence of approximate solutions to the exact solution.

Keywords: cordial integral equation, singular fractional differential equation, analytic solution, exponential convergence, collocation method.

AMS Subject Classification: 34A08; 45D05; 45J05; 47G20; 65R20.

1 Introduction

Cordial Volterra integral equations of the second kind have been considered recently in a number of papers, e.g. [3], [6], [7]. These equations are usually in the form

$$\mu u(t) = \int_0^t \frac{1}{t}\varphi\left(\frac{s}{t}\right) a(t,s)u(s)ds + f(t), \quad 0 \le t \le T \tag{1.1}$$

or in a slightly more general form

$$\mu u(t) = \int_0^1 \varphi(x)b(t,x)u(tx)dx + f(t), \quad 0 \le t \le T. \tag{1.2}$$

Here $\varphi \in L^1(0,1)$ is called the core of the corresponding cordial integral operator, a, b and f are given smooth enough functions. We discuss under which

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conditions these equations admit solutions, which are analytic in some open region around $[0, T]$. We also discuss equations which can contain in addition to the one cordial integral operators, other cordial or ordinary Volterra integral operators. We show that certain singular fractional differential or integro-differential equations can be reduced to cordial integral equations and prove existence of analytic solutions of these equations. We also present a numerial scheme for these equations, which has exponential rate of convergence. Cordial integral equations of the form (1.1), with just one integral operator, are discussed in [3, 6, 7] and singular fractional differential equations without the integral term are considered in [2, 4].

In Section 2 we introduce a more general form of cordial Volterra integral equation and also introduce singular fractional differential equations, which can be reduced to this more general form. All later results about existence and uniqueness of the solutions and about the convergence of the numerical schemes apply to these general equations.

In Section 3 we study properties of the cordial integral operator in spaces of analytic functions. It is known (see [6], [7]) that the operator in (1.1) is not compact in spaces $C^m[0, T]$, if $a(0, 0) \neq 0$. In contrast, it turns out that in spaces of functions, which are analytic in some region around $[0, T]$, the operators in (1.1) and (1.2) are compact. In spaces of analytic functions compactness of the cordial integral operator was proved in [3]. We describe exactly the spectrum of the sum of cordial integral operators in the spaces of analytic functions. In contrast, in [3] it was proved that the spectrum of the operator belongs to a certain set; here we prove that the spectrum is exactly equal to this set, and we also prove that the spectrum of a sum of cordial integral operators consists actually of just the sum of the corresponding eigenvalues, which is quite surprising, because the eigenspaces are not the same. Similar results have been obtained in [2] in spaces C^m and in [4] in spaces of analytic functions, but only for cores of a very specific type; here we prove the result for general cores and for general coefficients b.

In Section 4 we derive the results about the existence and uniqueness of the solution in spaces of analytic functions. Special cases of these results are given in [3, 4]. We point out that even when the solution is unique in these spaces, it may not be unique in some wider space, e.g. $C^m[0, T]$.

The analyticity of the solution means that it is possible to construct methods of solution for these equations which have exponential convergence in the number of parameters in discretized equation. In Section 5 we propose the polynomial collocation method with Chebyshev nodes as the collocation points and prove exponential convergence of this method. We solve the discrete equations only on $[0, T]$, but the approximate solutions converge exponentially to the exact solution in some wider region of the complex plane. Similar method has been discussed in [7] in spaces $C^m[0, T]$, and in [3, 4] in spaces of analytic functions. Compared to results in [3, 4], the equations considered are more general, and we managed to remove one quite restrictive assumption on cores allowed and on orders of fractional differential equations, which has been a challenge for several years.

In [11] a cordial integro-differential equation has been solved, using polynomial collocation method with Chebyshev nodes. One of the numerical examples there shows exponential convergence, even though it was not proved theoretically.

Similar results can be obtained for cordial integral equations of the first kind. In [8] and [9] cordial integral equations of the first kind are studied in spaces $C^m[0,T]$ (or in more general weighted spaces). The main idea in these papers is to reduce the equation to an equation of the second kind and this can be done in spaces of analytic functions as well.

2 General equation

2.1 Generalized cordial Volterra integral operators

Let $\varphi \in L^1(0,1)$ and $a \in C(\Delta_T)$ be given, where $\Delta_T = \{(t,s) : 0 \le s \le t \le T\}$. Define the cordial integral operator $V_{\varphi,a}$ by

$$(V_{\varphi,a}u)(t) = \int_0^t \frac{1}{t}\varphi\left(\frac{s}{t}\right) a(t,s)u(s)ds, \quad 0 \le t \le T. \tag{2.1}$$

These operators have been studied thoroughly in spaces $C^m[0,T]$, case $a \equiv 1$ in [6] and the general case in [7]. In most cases we actually prove results for slightly more general integral operators $\widetilde{V}_{\varphi,b}$ defined by

$$\left(\widetilde{V}_{\varphi,b}u\right)(t) = \int_0^1 \varphi(x)b(t,x)u(tx)dx, \quad 0 \le t \le T, \tag{2.2}$$

which allows the results to hold for a more general class of operators. For any smooth a the operator in the form (2.1) can be written in the form (2.2) with smooth b by making a change of variables $s = xt$ in the integral (then $b(t,x) = a(t,xt)$), but not vice versa: smooth b generally corresponds to a, which is discontinuous at (0,0). In case $a \equiv 1$ (or $b \equiv 1$) we simply use the notation V_φ; in this case $V_\varphi = \widetilde{V}_\varphi$.

Note that the usual Volterra integral operator with continuous kernel

$$(Ku)(t) = \int_0^t k(t,s)u(s)ds$$

can also be written in the form (2.1) by choosing $\varphi(x) \equiv 1$ and $a(t,s) = tk(t,s)$. Also the Volterra integral operator with weakly singular kernel

$$(K_1u)(t) = \int_0^t (t-s)^{-\alpha}s^{-\beta}k(t,s)u(s)ds,$$

where k is continuous and $\alpha, \beta < 1$, $\alpha + \beta < 1$ can be written in the form (2.1) by choosing $\varphi(x) = (1-x)^{-\alpha}x^{-\beta}$ and $a(t,s) = t^{1-\alpha-\beta}k(t,s)$. Hence all results obtained for cordial integral equations in the special case $a(0,0) = 0$ or $b(0,x) \equiv 0$ apply to usual Volterra integral equations as well (these results are, of course well known already). In this sense the usual Volterra integral equations are a special case of cordial integral equations.

2.2 Singular fractional integro-differential equations

We consider fractional differential equations which have a singularity of certain type at 0. In addition, they can also contain an integral term. It turns out that these equations can also be reduced to a general cordial Volterra integral equation.

First we define the fractional derivative used here. The Riemann-Liouville fractional integral operator J^ν is given by

$$(J^\nu u)(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} u(s) ds, \quad u \in C[0,T],\ \nu > 0,\ t \geq 0$$

and $J^0 = I$, where Γ is the Euler Gamma function. We use fractional differential operator D_0^ν, $\nu \geq 0$, which is defined by $D_0^\nu = (J^\nu)^{-1}$ on $J^\nu C[0,T]$. Note that the space $J^\nu C[0,T]$ is described in [10]. Define also the multiplication operator M_α for $\alpha \in \mathbb{R}$ by $(M_\alpha u)(t) = t^\alpha u(t)$. We study singular fractional differential equations of the form

$$(D_0^\alpha M_\alpha u)(t) = \sum_{k=1}^{l} a_k(t)(D_0^{\alpha_k} M_{\alpha_k} u)(t) + a_0(t)(V_\varphi u)(t) + f(t),\ 0 < t \leq T, \tag{2.3}$$

where $\alpha > \alpha_k \geq 0$, a_k, f are given sufficiently regular functions and $u \in C[0,T]$ is unknown. The operator V_φ is a cordial Volterra integral operator with the core $\varphi \in L^1(0,1)$. Instead of operator $a_0 V_\varphi$ there can also be a sum of operators of this type, but we do not want to make the notation too cumbersome. Probably the results remain similar, if instead of $a_0 V_\varphi$ we have $V_{\varphi,a}$ or $\widetilde{V}_{\varphi,b}$, but it seems hard to prove that the product (or even the sum!) of two cordial operators is again a cordial operator in the sense of (2.1) or (2.2), and it is possible that this does not hold in general. Maybe the class of cordial integral operators should be made even more general. In case $a \equiv 1$ both the sum and product of cordial integral operators are also cordial; for the sum it is obvious and for the product see [6].

Note that for a smooth enough u the operator $D_0^\nu M_\nu$ can always be applied to u, because $M_\nu u \in J^\nu C[0,T]$. Actually instead of the operators D_0^ν the Riemann-Liouville or the Caputo derivative of order ν can be used, because when applied to functions of the form $t^\nu u(t)$ with smooth enough u they all give the same results.

These equations (without the integral term) have been studied in spaces $C^m[0,T]$ in [2]. Note that to get a unique smooth solution of (2.3), no initial or boundary condition are allowed: the smooth solution is unique, and imposing any additional conditions generally results in a nonsmooth solution. Fractional differential equations without singularities and with initial conditions are discussed e.g. in [1].

In the following we reduce the equation to a cordial Volterra integral equation, using the ideas from [2]. We make a change of variables $v = D_0^\alpha M_\alpha u$ in equation (2.3). Note that $u = (D_0^\alpha M_\alpha)^{-1} v = M_{-\alpha} J^\alpha v$, hence (2.3) is equivalent to

$$v = \sum_{k=1}^{l} a_k D_0^{\alpha_k} M_{\alpha_k} M_{-\alpha} J^\alpha v + a_0 V_\varphi M_{-\alpha} J^\alpha v + f. \tag{2.4}$$

Operator $M_{-\alpha}J^\alpha$ can be written as a cordial integral operator:

$$(M_{-\alpha}J^\alpha u)(t) = t^{-\alpha}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,ds = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{t}\left(1-\frac{s}{t}\right)^{\alpha-1} u(s)ds.$$

To write the operators $D_0^{\alpha_k} M_{\alpha_k} M_{-\alpha} J^\alpha$ as cordial integral operators one can first apply these operators to functions $w_n(t) = t^n$, $n = 0, 1, \ldots$, write the results in the form not depending on n, and then use density of polynomials in $C[0, T]$. This results in

$$(D_0^{\alpha_k} M_{\alpha_k} M_{-\alpha} J^\alpha u)(t) = \frac{1}{\Gamma(\alpha-\alpha_k)}\int_0^t \frac{1}{t}\left(1-\frac{s}{t}\right)^{\alpha-\alpha_k-1}\left(\frac{s}{t}\right)^{\alpha_k} u(s)\,ds.$$

The integral part of the equation (2.4) can be written as

$$\begin{aligned}(V_\varphi M_{-\alpha} J^\alpha v)(t) &= \int_0^1 \varphi(x) t^{-\alpha} x^{-\alpha} \frac{1}{\Gamma(\alpha)} \int_0^{xt} (xt-s)^{\alpha-1} v(s)\,ds\,dx \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{t}\int_{s/t}^1 x^{-\alpha}\left(x-s/t\right)^{\alpha-1}\varphi(x)dx\, v(s)\,ds,\end{aligned}$$

which is again a cordial integral operator with the core

$$\varphi_0(y) = \frac{1}{\Gamma(\alpha)}\int_y^1 x^{-\alpha}\left(x-y\right)^{\alpha-1}\varphi(x)dx. \tag{2.5}$$

It can be easily checked that $\varphi_0 \in L^1(0,1)$:

$$\begin{aligned}\int_0^1 |\varphi_0(y)|dy &\le \frac{1}{\Gamma(\alpha)}\int_0^1\int_y^1 x^{-\alpha}\left(x-y\right)^{\alpha-1}|\varphi(x)|dx\,dy \\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 |\varphi(x)|x^{-\alpha}\int_0^x \left(x-y\right)^{\alpha-1} dy\,dx = \frac{\|\varphi\|_{L^1}}{\alpha\Gamma(\alpha)}.\end{aligned}$$

Denote

$$\varphi_\alpha(x) = \frac{1}{\Gamma(\alpha)}(1-x)^{\alpha-1} \quad \text{and} \quad \varphi_{\alpha,\alpha_k}(x) = \frac{1}{\Gamma(\alpha-\alpha_k)}(1-x)^{\alpha-\alpha_k-1}x^{\alpha_k}.$$

Note that $\varphi_\alpha \in L^1(0,1)$ if and only if $\alpha > 0$, and $\varphi_{\alpha,\alpha_k} \in L^1(0,1)$ if and only if $\alpha > \alpha_k > -1$. Since for negative α_k the operator $D_0^{\alpha_k}$ is not defined, we assume $\alpha_k \ge 0$. Now the equation (2.4) can be written as

$$v = \sum_{k=1}^{l} a_k V_{\varphi_{\alpha,\alpha_k}} v + a_0 V_{\varphi_0} v + f. \tag{2.6}$$

After solving this equation for v we can get the solution of the equation (2.3) by $u = V_{\varphi_\alpha} v$.

2.3 General cordial Volterra integral equation

Since it is not obvious how to write a sum of cordial integral operators as one cordial integral operator, we consider general equations of the form

$$\mu u(t) = \sum_{k=0}^{n} (\widetilde{V}_{\varphi_k, b_k} u)(t) + f(t), \quad 0 < t \leq T. \tag{2.7}$$

Note that equation (2.6) is also of this form. We will discuss, under which conditions this equation has a unique analytic solution in some open region containing $[0, T]$.

3 Properties of cordial integral operators

In this section we study integral operators of the form (2.2) in the spaces of analytic functions.

Let D be a bounded open domain in the complex plane containing $[0, T]$ such that if $t \in D$, then $tx \in D$ $\forall x \in [0, 1]$ (that means D is star-shaped with respect to 0). Let $\mathcal{A}(D)$ be the space of functions analytic in D and continuous on $\overline{D}$ with the norm $\|v\|_\infty = \max_{t\in\overline{D}} |v(t)|$. In the following we actually consider operators $V_{\varphi,a}$ and $\widetilde{V}_{\varphi,b}$ acting in the space $\mathcal{A}(D)$. If $b(t, x)$ is defined for $t \in \overline{D}$ (or can be analytically extended to $\overline{D}$) then $(\widetilde{V}_{\varphi,b} u)(t)$ is defined for $t \in \overline{D}$. If the function $a(t, x)$ is defined for $t \in \overline{D}$ and for $s \in [0, t]$ with $t \in \overline{D}$ then $(V_{\varphi,a} u)(t)$ is also defined for $t \in \overline{D}$. In the following we show that if $u \in \mathcal{A}(D)$ then these extensions of $V_{\varphi,a} u$ and $\widetilde{V}_{\varphi,b} u$ under quite general conditions belong to $\mathcal{A}(D)$.

3.1 Compactness of cordial integral operator

First we show that for analytic b the operator $\widetilde{V}_{\varphi,b}$ can be considered in the space $\mathcal{A}(D)$ and it is compact in this space. This result is proved in [3], but for completeness we bring a sketch of proof here.

Theorem 1. *Let $\varphi \in L^1(0, 1)$, $b \in C(\overline{D} \times [0, 1])$ and $b(\cdot, x) \in \mathcal{A}(D)$ $\forall x \in [0, 1]$. Then $\widetilde{V}_{\varphi,b} \in \mathcal{K}(\mathcal{A}(D))$ and*

$$\|\widetilde{V}_{\varphi,b}\|_{\mathcal{A}(D)} \leq \|\varphi\|_{L^1} \|b\|_\infty.$$

Proof. Let $u \in \mathcal{A}(D)$ be given. Then $(\widetilde{V}_{\varphi,b} u)(t)$ is defined for all $t \in \overline{D}$ and is obviously continuous in $\overline{D}$. For any closed contour γ in D we have

$$\int_\gamma (\widetilde{V}_{\varphi,b} u)(t) dt = 0,$$

because the integral signs can be exchanged and $b(t, x)$ and $u(tx)$ are analytic in t for any $x \in [0, 1]$. Hence by Morera's theorem, $\widetilde{V}_{\varphi,b} u \in \mathcal{A}(D)$. The estimate for the norm is also straightforward.

To show compactness of $\widetilde{V}_{\varphi,b}$, we can approximate the kernel of $\widetilde{V}_{\varphi,b}$ by smoother functions and show compactness of these smoother operators. Details can be found in [3]. □

Similar result can be obtained for operator $V_{\varphi,a}$, but to get the most general result one actually has to make the change of variables $s = tx$. Let

$$G = \{(t,s) : t \in D,\ s \in [0,t]\}.$$

Corollary 1. Let $\varphi \in L^1(0,1)$, $a \in C(\overline{G})$ and $\forall x \in [0,1]$, $(t \mapsto a(t,tx)) \in \mathcal{A}(D)$. Then $V_{\varphi,a} \in \mathcal{K}(\mathcal{A}(D))$ and

$$\|V_{\varphi,a}\|_{\mathcal{A}(D)} \le \|\varphi\|_{L^1}\|a\|_\infty.$$

In contrast, $\widetilde{V}_{\varphi,b}$ is compact in $C^m[0,T]$ if and only if $b(0,x) = 0\ \forall x \in [0,1]$; $V_{\varphi,a}$ is compact in $C^m[0,T]$ if and only if $a(0,0) = 0$ (see [7] for $V_{\varphi,a}$; for $\widetilde{V}_{\varphi,b}$ the proofs are similar).

3.2 The spectrum of a cordial integral operator

For $\psi \in L^1(0,1)$ and $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \ge 0$, denote $\hat{\psi}(\lambda) = \displaystyle\int_0^1 x^\lambda \psi(x)dx.$

In the following let $\psi(x) = \varphi(x)b(0,x)$. It is known that if we consider $V_{\varphi,a}$ as acting in $C^m[0,T]$ (assuming a is m times continuously differentiable), then its spectrum is given by

$$\sigma_m(V_{\varphi,a}) = \{0\} \cup \{a(0,0)\hat{\varphi}(j),\ j = 0,\dots,m\} \cup \{a(0,0)\hat{\varphi}(\lambda),\ \operatorname{Re}\lambda \ge m\}$$

(see [7]). Similarly one can get that the spectrum of $\widetilde{V}_{\varphi,b}$ in $C^m[0,T]$ (assuming b is m times continuously differentiable with respect to t) is given by

$$\sigma_m(\widetilde{V}_{\varphi,b}) = \{0\} \cup \{\hat{\psi}(j),\ j = 0,\dots,m\} \cup \{\hat{\psi}(\lambda),\ \operatorname{Re}\lambda \ge m\}.$$

Let

$$\sigma_\infty(V_{\varphi,a}) = \{0\} \cup \{a(0,0)\hat{\varphi}(j), j=0,1,\dots\},\ \sigma_\infty(\widetilde{V}_{\varphi,b}) = \{0\} \cup \{\hat{\psi}(j), j=0,1,\dots\}.$$

Denote by $\sigma_{\mathcal{A}(D)}(V_{\varphi,a})$ and $\sigma_{\mathcal{A}(D)}(\widetilde{V}_{\varphi,b})$ the spectrums of $V_{\varphi,a}$ and $\widetilde{V}_{\varphi,b}$ in $\mathcal{A}(D)$. The exact description of $\sigma_{\mathcal{A}(D)}(\widetilde{V}_{\varphi,b})$ is given in the following theorem; the easy part of this theorem, namely $\sigma_{\mathcal{A}(D)}(\widetilde{V}_{\varphi,b}) \subset \sigma_\infty(\widetilde{V}_{\varphi,b})$ was proven in [3].

Theorem 2. *Under the assumptions of Theorem 1 we have*

$$\sigma_{\mathcal{A}(D)}(\widetilde{V}_{\varphi,b}) = \sigma_\infty(\widetilde{V}_{\varphi,b}).$$

Proof. Since $\widetilde{V}_{\varphi,b}$ is compact, its spectrum consists of 0 and eigenvalues of the operator. Since all eigenfunctions in $\mathcal{A}(D)$ also belong to $C^m[0,T]$ for any m, we have

$$\sigma_{\mathcal{A}(D)}(\widetilde{V}_{\varphi,b}) \subset \bigcap_{m=0}^{\infty} \sigma_m(\widetilde{V}_{\varphi,b}) = \sigma_\infty(\widetilde{V}_{\varphi,b}).$$

To get the inclusion $\sigma_\infty(\widetilde{V}_{\varphi,b}) \subset \sigma_{\mathcal{A}(D)}(\widetilde{V}_{\varphi,b})$ let $\lambda \in \sigma_\infty(\widetilde{V}_{\varphi,b}) \setminus \{0\}$ and let j be the last integer for which $\hat{\psi}(j) = \lambda$ (the last integer always exists, because $\lim_{n\to\infty} \hat{\psi}(n) = 0$). Look for solutions of $\widetilde{V}_{\varphi,b}u = \lambda u$ of the form $u(t) = t^j + t^{j+1}v(t)$. Then

$$\begin{aligned}\left(\widetilde{V}_{\varphi,b}u\right)(t) &= \int_0^1 \varphi(x)b(t,x)t^j x^j dx + \int_0^1 \varphi(x)b(t,x)t^{j+1}x^{j+1}v(tx)dx \\ &= \lambda t^j + t^j \int_0^1 \varphi(x)(b(t,x) - b(0,x))x^j dx + t^{j+1}\int_0^1 \varphi(x)b(t,x)x^{j+1}v(tx)dx.\end{aligned}$$

Hence v satisfies (after cancelling λt^j and dividing both sides of the equation by t^{j+1})

$$\lambda v(t) = \int_0^1 \varphi(x)b(t,x)x^{j+1}v(tx)dx + \int_0^1 \varphi(x)\frac{b(t,x) - b(0,x)}{t}x^j dx,$$

which can be written as $\lambda v = Wv + f$. Here W is a cordial integral operator and $f \in \mathcal{A}(D)$ (the apparent singularity of $\dfrac{b(t,x) - b(0,x)}{t}$ at $t = 0$ is removable). We know that W is compact and

$$\sigma_{\mathcal{A}(D)}(W) \subset \sigma_\infty(W) = \{0\} \cup \{\hat{\psi}(l),\ l = j+1, j+2, \ldots\},$$

hence λ does not belong to the spectrum of W, therefore the equation for v is uniquely solvable in $\mathcal{A}(D)$. Consequently $\lambda \in \sigma_{\mathcal{A}(D)}(\widetilde{V}_{\varphi,b})$. $\square$

Corollary 2. Under the assumptions of Corollary 1 we have

$$\sigma_{\mathcal{A}(D)}(V_{\varphi,a}) = \sigma_\infty(V_{\varphi,a}).$$

3.3 The spectrum of a sum of cordial integral operators

To get the existence and uniqueness results for equation (2.7), we have to study the spectrum of the sum of cordial integral equations. Assume b_k and φ_k satisfy the assumptions of Theorem 1. Let $\psi_k(x) = \varphi_k(x)b_k(0,x)$, $x \in (0,1)$, $k = 0, \ldots, n$ and denote

$$Wu = \sum_{k=0}^{n} \widetilde{V}_{\varphi_k,b_k}u \quad \text{and } W_0u = \sum_{k=0}^{n} V_{\psi_k}u. \tag{3.1}$$

Since in the space $\mathcal{A}(D)$ the eigenvalues of V_{ψ_k} are $\hat{\psi}_k(j)$ with the corresponding eigenfunctions t^j, $j = 0, 1, \ldots$ we immediately get

$$\sigma_{\mathcal{A}(D)}(W_0) = \{0\} \cup \{\sum_{k=0}^{n} \hat{\psi}_k(j),\ j = 0, 1, \ldots\}.$$

We show that the same result is true for W.

Theorem 3. *Let $\varphi_k \in L^1(0,1)$, $b_k \in C(\overline{D} \times [0,1])$ and $b_k(\cdot, x) \in \mathcal{A}(D)$ $\forall x \in [0,1]$, $k = 0, \dots, n$. Then*

$$\sigma_{\mathcal{A}(D)}(W) = \{0\} \cup \left\{ \sum_{k=0}^{n} \hat{\psi}_k(j),\ j = 0, 1, \dots \right\}.$$

Proof. First we will show that $\sigma_{\mathcal{A}(D)}(W) \subset \sigma_{\mathcal{A}(D)}(W_0)$. Note that

$$(W_0 u)^{(j)}(t) = \sum_{k=0}^{n} \int_0^1 \psi_k(x) x^j u^{(j)}(xt)dx,\ j = 0, 1, \dots,$$

hence

$$(W_0 u)^{(j)}(0) = \sum_{k=0}^{n} \hat{\psi}_k(j) u^{(j)}(0),\ j = 0, 1, \dots.$$

Suppose by contradiction that $\lambda \in \sigma_{\mathcal{A}(D)}(W)$, but $\lambda \notin \sigma_{\mathcal{A}(D)}(W_0)$. Then there exists $v \in \mathcal{A}(D)$, $v \not\equiv 0$ such that $\lambda v = Wv$. Since $(Wv - W_0 v)(0) = 0$, we have

$$\lambda v(0) = (Wv)(0) = (W_0 v)(0) = \sum_{k=0}^{n} \hat{\psi}_k(0) v(0).$$

Since $\lambda \neq \sum_{k=0}^{n} \hat{\psi}_k(0)$, it follows that $v(0) = 0$. Taking the derivative of $\lambda v = Wv$ and evaluating it at 0 we get

$$\lambda v'(0) = (Wv)'(0) = (W_0 v)'(0) + \sum_{k=0}^{n} (\widetilde{V}_{(b_k)_t, \varphi_k} v)(0) = \sum_{k=0}^{n} \hat{\psi}_k(1) v'(0).$$

Since $\lambda \neq \sum_{k=0}^{n} \hat{\psi}_k(1)$, it follows that $v'(0) = 0$. Continuing taking derivatives, we always get (using the fact that the lower derivatives of v at 0 are zero)

$$\lambda v^{(j)}(0) = (Wv)^{(j)}(0) = (W_0 v)^{(j)}(0) = \sum_{k=0}^{n} \hat{\psi}_k(j) v^{(j)}(0),$$

hence all derivatives of v at 0 must be 0. Since v is analytic, this implies that $v \equiv 0$, a contradiction.

To show that $\sigma_{\mathcal{A}(D)}(W_0) \subset \sigma_{\mathcal{A}(D)}(W)$ we use the same idea as in proving Theorem 2. Let $\lambda \in \sigma_{\mathcal{A}(D)}(W_0) \setminus \{0\}$ and let j be the last integer for which $\sum_{k=0}^{n} \hat{\psi}_k(j) = \lambda$. Look for solutions of $Wu = \lambda u$ (eigenfunctions of W) of the form $u(t) = t^j + t^{j+1} v(t)$. Substituting this into the equation, writing $\varphi_k(x) b_k(t, x) = \psi_k(x) + \varphi_k(x)(b_k(t, x) - b_k(0, x))$, cancelling λt^j and dividing both sides of the equation by t^{j+1} we conclude that v satisfies

$$\lambda v(t) = \sum_{k=0}^{n} \int_0^1 \varphi_k(x) b_k(t, x) x^{j+1} v(xt)dx + \sum_{k=0}^{n} \int_0^1 \varphi_k(x) \frac{b_k(t, x) - b_k(0, x)}{t} x^j dx,$$

which can be written as $\lambda v = \widetilde{W}_j v + f$, where $\widetilde{W}_j$ has the same form as W and $f \in \mathcal{A}(D)$. By the first part of the proof we already know that

$$\sigma_{\mathcal{A}(D)}(\widetilde{W}_j) \subset \{0\} \cup \{\hat{\psi}(l),\ l = j+1, j+2, \ldots\},$$

hence λ does not belong to the spectrum of $\widetilde{W}_j$ and hence the equation for v is uniquely solvable in $\mathcal{A}(D)$. Consequently $\lambda \in \sigma_{\mathcal{A}(D)}(W)$. $\square$

Corollary 3. The spectrum of the operator $W = \sum_{k=1}^{l} a_k V_{\varphi_{\alpha,\alpha_k}} + a_0 V_{\varphi_0}$ in equation (2.6), with φ_0 defined by (2.5), is given by

$$\sigma_{\mathcal{A}(D)}(W) = \{0\} \cup \left\{ a_0(0) \frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)} \hat{\varphi}(j) + \sum_{k=1}^{l} a_k(0) \frac{\Gamma(j+\alpha_k+1)}{\Gamma(j+\alpha+1)},\ j=0,1,\ldots \right\}.$$

Proof. Just use Theorem 3 and note that

$$\hat{\varphi}_{\alpha,\alpha_k}(j) = \frac{1}{\Gamma(\alpha-\alpha_k)} \int_0^1 x^j (1-x)^{\alpha-\alpha_k-1} x^{\alpha_k} dx = \frac{\Gamma(j+\alpha_k+1)}{\Gamma(j+\alpha+1)}$$

and

$$\begin{aligned} \hat{\varphi}_0(j) &= \frac{1}{\Gamma(\alpha)} \int_0^1 y^j \int_y^1 x^{-\alpha} (x-y)^{\alpha-1} \varphi(x) dx\, dy \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 \int_0^x y^j (x-y)^{\alpha-1} dy\, x^{-\alpha} \varphi(x) dx = \frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)} \hat{\varphi}(j). \end{aligned}$$

$\square$

4 Existence and uniqueness of analytic solutions

4.1 Existence and uniqueness for general cordial Volterra integral equation

Now we turn to the question when equation (2.7) admits an analytic solution. We note that the equation can still be considered only for $t \in (0, T]$, then the results mean that the solution can be analytically extended to some region in the complex plane. On the other hand, if a, b and f are analytic, the extensions actually satisfy the equation with t being in this larger region.

The main result is the following.

Theorem 4. *Let $\varphi_k \in L^1(0,1)$, $b_k \in C(\overline{D} \times [0,1])$ and $b_k(\cdot, x) \in \mathcal{A}(D)$ $\forall x \in [0,1]$, $k = 0, \ldots, n$. Let $f \in \mathcal{A}(D)$ be given. Let W be defined by* (3.1) *and assume that $\mu \notin \sigma_{\mathcal{A}(D)}(W)$. Then equation (2.7) has a unique solution in $\mathcal{A}(D)$.*

Proof. Since $\mu I - \widetilde{V}_{\varphi,b}$ ia a Fredholm operator of index 0 and μ does not belong to the spectrum, the claim follows from Fredholm Alternative. $\square$

Similar result can be stated if the operators in (2.7) are of the form (2.1).

We note the solution may not be unique in some wider space, e.g. in $C^m[0,T]$, because μ may still be an eigenvalue of the operator in the wider space. We point out that under the assumptions of the theorem it is not possible to have solutions which belong to $C^\infty[0,T]$, but do not belong to $\mathcal{A}(D)$. This was an open problem stated in [3,4]; now it follows from the exact description of the spectrum, Theorems 2 and 3.

4.2 Existence and uniqueness for singular fractional integro-differential equation

For equation (2.6) we can directly use Theorem 4 to get the conditions which guarantee the existence and uniqueness of v. Under these conditions the existence of solution of the singular fractional integro-differential equation (2.3) follows from $u = V_{\varphi_\alpha} v$; the uniqueness of u follows from the fact that if u_1 and u_2 are analytic solutions of (2.3), then $v = D_0^\alpha M_\alpha(u_1 - u_2)$ is also analytic in some region containing 0, since $D_0^\alpha M_\alpha$ can be applied to a power series around 0 and does not change the radius of convergence of the power series. Since v is the unique solution of homogeneous equation (2.6) (possibly in a smaller region), it must be zero, and by uniqueness and analyticity of the solution of (2.6) in the larger region, the solution must be zero in the larger region as well. Hence $u_1 - u_2$ must also be zero.

The spectrum of the operator in (2.6) is described in Corollary 3. The existence and uniqueness result now follows easily.

Theorem 5. *Let $\alpha > \alpha_k \geq 0$, $a_k \in \mathcal{A}(D)$, $k = 0,\dots,l$, $\varphi \in L^1(0,1)$ and $f \in \mathcal{A}(D)$ be given. Assume that*

$$a_0(0)\frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)}\hat{\varphi}(j) + \sum_{k=1}^{l} a_k(0)\frac{\Gamma(j+\alpha_k+1)}{\Gamma(j+\alpha+1)} \neq 1, \quad j = 0,1,\dots.$$

Then equation (2.6) has a unique solution $v \in \mathcal{A}(D)$, and equation (2.3) has a unique solution $u = V_{\varphi_\alpha} v \in \mathcal{A}(D)$.

Remark 1. Obviously this result can also be extended to the case when in the fractional integro-differential equation (2.3) instead of one cordial integral operator we have a sum of cordial integral operators. We can just use Theorem 3 to get the spectrum of the corresponding sum of cordial operators.

5 Discretization

5.1 Polynomial collocation method

The analyticity of the solution means that it is possible to construct methods of solution for these equations which have exponential convergence in the number of free parameters. We propose to use the polynomial collocation method, where the collocation points are the Chebyshev points

$$t_j = \frac{T}{2}\left(1 - \cos\frac{(2j+1)\pi}{2(N+1)}\right),\ j = 0,\dots,N.$$

We look for solutions of (2.7) in the form $u_N(t) = \sum_{p=0}^{N} c_p t^p$. Then the collocation equations are

$$\sum_{p=0}^{N} c_p t_j^p \left(\mu - \sum_{k=0}^{n} \int_0^1 \varphi_k(x) b_k(t_j, x) x^p dx \right) = f(t_j), \quad j = 0, \dots, N. \tag{5.1}$$

To set up the system, one has to calculate exactly or "well enough" the integrals

$$\int_0^1 \varphi_k(x) b_k(t_j, x) x^p dx.$$

For theoretical results it is easiest to use the basis $\{t^p\}$ for polynomials; for practical calculations though, this results in very badly conditioned systems. So for larger N one has to use a better basis, for example the (scaled) Chebyshev polynomials $T_p(t) = \cos\left(p \arccos\left(\frac{2t}{T} - 1\right)\right)$. In fact, it may be simpler to make first the change of variables $t = \frac{T}{2}(1 - \cos s)$ and then work with trigonometric polynomials in s instead.

For the singular fractional integro-differential equation (2.3) the collocation equations are

$$\sum_{p=0}^{N} c_p t_j^p \left(\frac{\Gamma(\alpha + p + 1)}{\Gamma(p+1)} - \sum_{k=1}^{l} a_k(t_j) \frac{\Gamma(\alpha_k + p + 1)}{\Gamma(p+1)} - a_0(t_j)\hat{\varphi}(p) \right)$$
$$= f(t_j), \quad j = 0, \dots, N. \tag{5.2}$$

Note that if we first solve equation (2.6) for v using the polynomial collocation method with Chebyshev points as the collocation points and then calculate $u = V_{\varphi_\alpha} v$, then we end up getting the same approximate solution as when solving (2.3) directly. Therefore convergence results for collocation method (5.2) for (2.3) follow from the convergence results of this collocation method (5.1) for cordial integral equations.

5.2 Properties of the interpolation operator

Let $r > 1$ and let E_r be an ellipse with foci 0 and T and semiaxes $\frac{T}{2}\frac{r + r^{-1}}{2}$ and $\frac{T}{2}\frac{r - r^{-1}}{2}$. Let t_j be the Chebyshev points and let Q_N be the corresponding Lagrange interpolation operator. The convergence of the interpolation process in $[0, T]$ is a classical result; the convergence in ellipses is proved e.g. in [3], but since the proof there contains several errors and typos, we provide a new proof here.

Theorem 6. *Let $u \in \mathcal{A}(E_r)$ be given. Then*

$$\|Q_N u - u\|_{\mathcal{A}(E_\rho)} \le C_\rho \left(\frac{r}{\rho}\right)^{-N} \|u\|_{\mathcal{A}(E_r)}$$

for any $1 < \rho < r$*; in particular (corresponding to* $\rho = 1$*)*

$$\|Q_N u - u\|_{C[0,T]} \le C r^{-N} \|u\|_{\mathcal{A}(E_r)}.$$

Proof. Make a change of variables $t = \frac{T}{2}(1 - \cos s)$. Then $t \in E_r \Leftrightarrow s \in S_r$, where $S_r = \{s \in \mathbf{C} : |\mathrm{Im}\, s| < \ln r\}$. Let $v(s) = u\left(\frac{T}{2}(1 - \cos s)\right)$. Then $u \in C(\overline{E_r}) \Rightarrow v \in C(\overline{S_r})$, v is even and 2π-periodic and

$$(Q_N u)\left(\frac{T}{2}(1 - \cos s)\right) = (\varPi_N v)(s),$$

where $\varPi_N$ is the trigonometric interpolation operator of order N for even functions with interpolation nodes $s_j = \frac{(2j+1)\pi}{2(N+1)}$, $j = 0, \dots, N$.

Let $v(s) = \dfrac{v_0}{2} + \displaystyle\sum_{n=1}^{\infty} v_n \cos ns$ and let $w(x) = v(x + i \ln r) + v(x - i \ln r)$. Then w is continuous, even and 2π-periodic. We have $v_n = \dfrac{w_n}{r^n + r^{-n}}$, where $w(x) = \dfrac{w_0}{2} + \displaystyle\sum_{n=1}^{\infty} w_n \cos nx$.

Since for $n \ge N + 1$ we have

$$|(\varPi_N(\cos n\cdot))(s) - \cos ns| \le \rho^n + \rho^{-n} \text{ for } s \in \overline{S_\rho},$$

we obtain the estimate

$$\begin{aligned} \|(\varPi_N v)(s) - v(s)\|_{\mathcal{A}(S_\rho)} &\le \sum_{n=N+1}^{\infty} |w_n| \frac{\rho^n + \rho^{-n}}{r^n + r^{-n}} \le \sum_{n=N+1}^{\infty} |w_n| \frac{2\rho^n}{r^n} \\ &\le \frac{2\rho}{\sqrt{r^2 - \rho^2}} \sqrt{\frac{2}{\pi}} \left(\frac{r}{\rho}\right)^{-N} \|w\|_{L^2(0,\pi)} \le \frac{2\sqrt{2}\rho}{\sqrt{r^2 - \rho^2}} \left(\frac{r}{\rho}\right)^{-N} \|v\|_{\mathcal{A}(S_r)}. \end{aligned} \quad (5.3)$$

This together with $\|v\|_{\mathcal{A}(S_\rho)} = \|u\|_{\mathcal{A}(E_\rho)}$ gives us the first estimate of the theorem. To get the second one, just use $\rho = 1$ and the norms of $C[0, \pi]$ and $C[0, T]$ in the arguments above. □

Remark 2. From the proof one can see that the smoothness assumptions for u in the theorem can be weakened slightly: it is enough to assume that u is analytic in E_r and $u|_\varGamma \in L^2(\varGamma)$, where $\varGamma$ is the boundary of E_r (the norm $\|u\|_{L^2(\varGamma)}$ should then be used at the right hand side).

5.3 Convergence

To prove results about the convergence of the method we use a general convergence theorem for projection methods for equations of second kind (see [5], Theorem 13.9).

Theorem 7. *Let X be a Banach space, $A : X \to X$ be compact and $I - A$ be injective. Assume that the projection operators $P_N : X \to X_N$ satisfy $\|P_N A - A\| \to 0$ as $N \to \infty$. Then for N large enough the approximate equation*

$$u_N - P_N A u_N = P_N f$$

is uniquely solvable for all $f \in X$ and there holds an error estimate

$$\|u_N - u\| \le M \|P_N u - u\|$$

with some positive constant M depending only on A. Here u is the unique solution of $u = Au + f$.

Now we can prove the exponential convergence of the method.

Theorem 8. *Let $\varphi_k \in L^1(0,1)$, $b_k \in C(\overline{E_r} \times [0,1])$ and $b_k(\cdot, x) \in \mathcal{A}(E_r)$ $\forall x \in [0,1]$, $k = 0, \dots, n$. Assume that $\dfrac{\varphi_k(x)}{\sqrt{1-x}} \in L^1(0,1)$, $k = 0, \dots, n$. Let $f \in \mathcal{A}(E_r)$ be given. Let W be defined by (3.1) and assume that $\mu \notin \sigma_{\mathcal{A}(E_r)}(W)$. Then for N large enough the collocation system (5.1) is uniquely solvable and its solution u_N satisfies*

$$\|u_N - u\|_{\mathcal{A}(E_\rho)} \le C_\rho \left(\frac{r}{\rho}\right)^{-N} \|u\|_{\mathcal{A}(E_r)} \quad \textit{for any } 1 < \rho < r,$$

where u is the unique solution of (2.7).

Proof. We use Theorem 7 with $X = \mathcal{A}(E_\rho)$, $A = W = \sum_{k=0}^{n} \widetilde{V}_{\varphi_k, b_k}$ and $P_N = Q_N$. Usually one shows that $\|P_N A - A\| \to 0$ by showing that A maps u to some "better" (smoother) space; here this approach does not seem to work, at least if we do not want to make additional assumptions about smoothness of φ.

Let $u \in \mathcal{A}(E_\rho)$. Let $l_j(t)$, $j = 0, \dots, N$ be the Lagrange interpolation polynomials for the Chebyshev nodes. We calculate first the difference for just one term of the sum:

$$\begin{aligned}
&(Q_N \widetilde{V}_{\varphi_k, b_k} u)(t) - (\widetilde{V}_{\varphi_k, b_k} u)(t) \\
&\quad = \sum_{j=0}^{N} \int_0^1 \varphi_k(x) b_k(t_j, x) u(t_j x) dx\, l_j(t) - \int_0^1 \varphi_k(x) b_k(t, x) u(tx) dx \\
&\quad = \int_0^1 \varphi_k(x) \Big(\sum_{j=0}^{N} b_k(t_j, x) u(t_j x) l_j(t) - b_k(t, x) u(tx) \Big) dx \\
&\quad = \int_0^1 \varphi_k(x) \left((Q_N (b_k(\cdot, x) u(\cdot x)))(t) - b_k(t, x) u(tx) \right) dx.
\end{aligned}$$

For fixed $x \in (0,1)$, $u(\cdot x) \in \mathcal{A}(E_{\rho(x)})$, where $\rho(x)$ is such that $\forall t \in E_{\rho(x)}$, $xt \in E_\rho$. This gives

$$\rho(x) = \left(\rho - 1 + \sqrt{(\rho-1)^2 + 4\rho x}\right)^2 / (4\rho x).$$

Since $b(\cdot,x)\in\mathcal{A}(E_r)$, by choosing $r(x)=\min\{\rho(x),r\}$ we get from the estimate (5.3) in the end of Theorem 6

$$\|Q_N(b(\cdot,x)u(\cdot x)-b(\cdot,x)u(\cdot x)\|_{\mathcal{A}(E_\rho)} \le \frac{2\sqrt{2}\rho}{\sqrt{r(x)^2-\rho^2}}\left(\frac{r(x)}{\rho}\right)^{-N}\|b_k(\cdot,x)\|_{\mathcal{A}(E_r)}\|u\|_{\mathcal{A}(E_\rho)}.$$

Since $\rho(x)$ is decreasing in x, $\rho(x)\to\infty$ as $x\to 0+$ and $\rho(1)=\rho$, there exists $x_0<1$ such that $r(x)=r$ for $x\le x_0$ and $r(x)=\rho(x)$ for $x_0<x\le 1$. Since $\rho'(1)=-\dfrac{\rho(\rho-1)}{\rho+1}$ and $\rho''(x)>0$ for $x\in(0,1]$, we have $\rho(x)\ge\rho+\dfrac{\rho(\rho-1)}{\rho+1}(1-x)$, hence $\dfrac{\rho(x)}{\rho}>1$, if $x<1$, and $\rho(x)^2-\rho^2\ge\dfrac{2\rho^2(\rho-1)}{\rho+1}(1-x)$. Now we can estimate

$$\|Q_N\widetilde{V}_{\varphi_k,b_k}u-\widetilde{V}_{\varphi_k,b_k}u\|_{\mathcal{A}(E_\rho)}\le\int_0^1|\varphi_k(x)|\frac{2\sqrt{2}\rho}{\sqrt{r(x)^2-\rho^2}}\left(\frac{r(x)}{\rho}\right)^{-N}$$
$$\times\|b_k(\cdot,x)\|_{\mathcal{A}(E_r)}\|u(\cdot x)\|_{\mathcal{A}(E_{\rho(x)})}dx\le 2\sqrt{2}\rho\left(\int_0^{x_0}|\varphi_k(x)|\frac{1}{\sqrt{r^2-\rho^2}}\left(\frac{r}{\rho}\right)^{-N}dx\right.$$
$$\left.+\int_{x_0}^1\frac{|\varphi_k(x)|}{\sqrt{1-x}}\sqrt{\frac{\rho+1}{2\rho^2(\rho-1)}}\left(\frac{\rho(x)}{\rho}\right)^{-N}dx\right)\|b_k\|_\infty\|u\|_{\mathcal{A}(E_\rho)}.$$

Here the first term decays exponentially, and since $(\rho(x)/\rho)^{-N}\to 0$ for all $x\in(0,1)$, by Lebesgue's Dominated Convergence Theorem the second integral also converges to zero. Therefore, $\|Q_NW-W\|\to 0$ as $N\to\infty$.

Consequently the assumptions of Theorem 7 are satisfied and hence for N large enough the collocation system (5.1) is uniquely solvable and its solution u_N satisfies

$$\|u_N-u\|_{\mathcal{A}(E_\rho)}\le CM_\rho\|Q_Nu-u\|_{\mathcal{A}(E_\rho)}.$$

The error estimate in $\mathcal{A}(E_\rho)$ now follows from Theorem 6. $\square$

Remark 3. The assumptions $\dfrac{\varphi_k(x)}{\sqrt{1-x}}\in L^1(0,1)$ can probably be weakened, but it is not clear right now, what the optimal assumptions are.

This approach does not work directly for getting the error estimate in $C(0,T]$. Theorem 7 cannot be used in $C(0,T]$, since the operator W is not compact there. Instead we can choose some small $\varepsilon>0$ and use Theorem 8 in E_ρ with $\rho=\dfrac{r}{r-\varepsilon}$ and the fact that $\|u_N-u\|_{C[0,T]}\le\|u_N-u\|_{\mathcal{A}(E_\rho)}$ to get the following estimate.

Corollary 4. Under the assumptions of Theorem 8 we have

$$\|u_N-u\|_{C[0,T]}\le C_\varepsilon(r-\varepsilon)^{-N}\|u\|_{\mathcal{A}(E_r)}$$

for any small $\varepsilon>0$.

In fact, to get a better error estimate in $C(0,T]$ (without ε), we can write the equation as

$$\mu u = W_0 + (W - W_0)u + f$$

and use the fact that the space of polynomials of fixed order is an invariant subspace for W_0, and similar reasoning as in [7] to show that $W - W_0$ is compact in $C[0,T]$ and actually smoothing. This is shown for operators of type (2.1) in [7], but we expect that the reasoning works for more general operators as well.

To get convergence result for the singular fractional integro-differential equation (2.3) we rewrite the equation in the form (2.6) and then use Theorem 8.

Theorem 9. *Let $\alpha_k \geq 0$, $\alpha > \alpha_k + 1/2$, (if there are no lower order derivatives, then $\alpha > 1/2$), $a_k \in \mathcal{A}(E_r)$, $k = 0, \ldots, l$ and $f \in \mathcal{A}(E_r)$ be given. Assume that*

$$a_0(0)\frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)}\hat{\varphi}(j) + \sum_{k=1}^{l} a_k(0)\frac{\Gamma(j+\alpha_k+1)}{\Gamma(j+\alpha+1)} \neq 1, \quad j = 0, 1, \ldots.$$

Let u be the unique solution of (2.3). Then for N large enough the collocation system (5.1) is uniquely solvable and its solution u_N satisfies

$$\|u_N - u\|_{\mathcal{A}(E_\rho)} \leq C\,(r/\rho)^{-N}\,\|u\|_{\mathcal{A}(E_r)} \quad \text{for any } 1 < \rho < r.$$

In particular,

$$\|u_N - u\|_{C[0,T]} \leq C_\varepsilon (r-\varepsilon)^{-N} \|u\|_{\mathcal{A}(E_r)}.$$

Proof. We need to show that the operators in (2.6) satisfy the conditions of Theorem 8. Since $\alpha > \alpha_k + 1/2$, we have $\dfrac{\varphi_{\alpha,\alpha_k}(x)}{\sqrt{1-x}} \in L^1(0,1)$. Check that φ_0 also satisfies this condition:

$$\begin{aligned}
\int_0^1 \frac{|\varphi_0(y)|}{\sqrt{1-y}}dy &\leq \frac{1}{\Gamma(\alpha)}\int_0^1 \frac{1}{\sqrt{1-y}}\int_y^1 x^{-\alpha}\,(x-y)^{\alpha-1}\,|\varphi(x)|dx\,dy \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 |\varphi(x)|x^{-\alpha}\int_0^x (1-y)^{-1/2}\,(x-y)^{\alpha-1}\,dy\,dx \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 |\varphi(x)|\int_0^1 (1-xs)^{-1/2}\,(1-s)^{\alpha-1}\,ds\,dx < \frac{\|\varphi\|_{L^1}}{(\alpha-1/2)\Gamma(\alpha)},
\end{aligned}$$

if $\alpha > 1/2$.

Now we can use Theorem 8 for equation (2.6) to get the convergence of v_N to v and then use $u = V_{\varphi_\alpha}v$ and $u_N = V_{\varphi_\alpha}v_N$. □

6 Numerical examples

6.1 Numerical solution of singular fractional differential equations

Here we consider two very similar examples:

$$D_0^\alpha t^\alpha u(t) = u(t) + \sqrt{2-t}, \quad t \in [0,1] \tag{6.1}$$

with $\alpha = 3/2$ and $\alpha = 1/2$. The exact solution can be written as

$$u(t) = \frac{\sqrt{2}}{\Gamma(\alpha+1)-1} - \sqrt{2}\sum_{k=1}^{\infty} \frac{\Gamma(k-1/2)\, t^k}{2^{k+1}\sqrt{\pi}\,(\Gamma(\alpha+1)-\Gamma(k+1))}.$$

Here f has a singularity at $t = 2$ and hence $f \in \mathcal{A}(E_r)$, where $r = 3+2\sqrt{2} \approx 5.83$. Since $\alpha_1 = 0$, the conditions of Theorem 9 require $\alpha > 1/2$. In the first case the condition is satisfied, but in the second case it is violated.

We used $N = 2, 3, 4, \ldots, 16$ and calculated the errors in the maximum norm on $t \in [0, 1]$ and the ratios of the corresponding errors. If the convergence is exponential, the ratios should be approximately 5.83. The results are presented in the following table.

Table 1. Errors and ratios of errors for equation (6.1) with $\alpha = 3/2$ and $\alpha = 1/2$.

N	error ($\alpha = 3/2$)	ratio	error ($\alpha = 1/2$)	ratio
2	$1.85 \cdot 10^{-3}$		$1.53 \cdot 10^{-2}$	
3	$1.93 \cdot 10^{-4}$	9.59	$1.82 \cdot 10^{-3}$	8.37
4	$2.28 \cdot 10^{-5}$	8.48	$2.35 \cdot 10^{-4}$	7.74
5	$2.89 \cdot 10^{-6}$	7.88	$3.26 \cdot 10^{-5}$	7.23
6	$3.86 \cdot 10^{-7}$	7.49	$4.64 \cdot 10^{-6}$	7.03
7	$5.34 \cdot 10^{-8}$	7.23	$6.83 \cdot 10^{-7}$	6.79
8	$7.59 \cdot 10^{-9}$	7.04	$1.02 \cdot 10^{-7}$	6.70
9	$1.10 \cdot 10^{-9}$	6.89	$1.55 \cdot 10^{-8}$	6.56
10	$1.62 \cdot 10^{-10}$	6.78	$2.39 \cdot 10^{-9}$	6.51
11	$2.43 \cdot 10^{-11}$	6.69	$3.72 \cdot 10^{-10}$	6.42
12	$3.68 \cdot 10^{-12}$	6.61	$5.81 \cdot 10^{-11}$	6.39
13	$5.61 \cdot 10^{-13}$	6.55	$9.19 \cdot 10^{-12}$	6.32
14	$8.52 \cdot 10^{-14}$	6.58	$1.45 \cdot 10^{-12}$	6.32
15	$1.33 \cdot 10^{-14}$	6.4	$2.50 \cdot 10^{-13}$	5.81
16	$3.55 \cdot 10^{-15}$	3.75	$3.91 \cdot 10^{-14}$	6.41

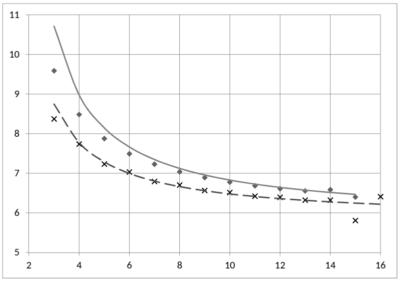

Figure 1. Ratios of the errors in cases $\alpha = 3/2$ (W) and $\alpha = 1/2$ ($\times$) and theoretical ratios if the convergence rate was $CN^{-\beta}r^N$ for $\beta = 3/2$ (continuous line) and $\beta = 1$ (dashed line).

The convergence rate is somewhat better than the theoretical estimate. Since the solution is actually better than just in $\mathcal{A}(E_r)$, we may expect the error to behave like $CN^{-\beta}r^{-N}$ for some $\beta > 0$. If the errors behave as

$CN^{-\beta}r^{-N}$, then the ratios should be $r\left(\frac{N}{N-1}\right)^{\beta}$. In Figure 1 the ratios of the errors are shown together with the theoretical ratios if the convergence rate was $CN^{-\beta}r^{-N}N$ for $\beta = 3/2$ and $\beta = 1$. It looks like the convergence rate is actually in the first case $N^{-3/2}r^{-N}$ and in the second case $N^{-1}r^{-N}$.

6.2 Numerical solution of singular fractional integro-differential equations

We consider the equation

$$D_0^{3/2}t^{3/2}u(t) + 3\sqrt{\pi}D(tu(t)) - \frac{3\sqrt{\pi}}{4}\int_0^t \frac{1}{t}u(s)\,ds$$
$$= \sqrt{1+t} + \frac{1}{t}\ln(1+t) + 2 + 2\sqrt{t}\arctan\sqrt{t}, \quad t \in [0,T] \quad (6.2)$$

on three different intervals: $T = 1, 2, 3$. The exact solution is

$$u(t) = \frac{(t+1)^2}{2t^{3/2}\sqrt{\pi}}\arctan\sqrt{t} + \frac{t-1}{2\sqrt{\pi t}}.$$

Both u and f have a removable singularity at $t = 0$ and a branching type of singularity at $t = -1$, so both of them can be continued to an analytic function on the whole complex plane, except a cut along the negative part of the real axis from -1 to $-\infty$. Hence $f \in \mathcal{A}(E_r)$, where $\frac{T}{2}\left(1 - \frac{1}{2}\left(r + \frac{1}{r}\right)\right) = -1$. For $T = 1$ this gives $r = 3 + 2\sqrt{2} \approx 5.83$, for $T = 2$ we have $r = 2 + \sqrt{3} \approx 3.73$ and for $T = 3$ we have $r = 3$.

Table 2. Errors and ratios for equation (6.2) with $T = 1$, $T = 2$ and $T = 3$.

N	error ($T = 1$)	ratio	error ($T = 2$)	ratio	error ($T = 3$)	ratio
2	$5.14 \cdot 10^{-4}$		$2.80 \cdot 10^{-3}$		$6.68 \cdot 10^{-3}$	
3	$5.17 \cdot 10^{-5}$	9.94	$4.56 \cdot 10^{-4}$	6.13	$1.38 \cdot 10^{-3}$	4.84
4	$5.79 \cdot 10^{-6}$	8.93	$8.26 \cdot 10^{-5}$	5.52	$3.16 \cdot 10^{-4}$	4.37
5	$6.94 \cdot 10^{-7}$	8.34	$1.60 \cdot 10^{-5}$	5.17	$7.70 \cdot 10^{-5}$	4.10
6	$8.72 \cdot 10^{-8}$	7.97	$3.23 \cdot 10^{-6}$	4.94	$1.96 \cdot 10^{-5}$	3.92
7	$1.13 \cdot 10^{-8}$	7.70	$6.77 \cdot 10^{-7}$	4.78	$5.17 \cdot 10^{-6}$	3.80
8	$1.51 \cdot 10^{-9}$	7.51	$1.45 \cdot 10^{-7}$	4.66	$1.40 \cdot 10^{-6}$	3.70
9	$2.05 \cdot 10^{-10}$	7.36	$3.18 \cdot 10^{-8}$	4.57	$3.84 \cdot 10^{-7}$	3.63
10	$2.82 \cdot 10^{-11}$	7.25	$7.09 \cdot 10^{-9}$	4.49	$1.08 \cdot 10^{-7}$	3.57
11	$3.93 \cdot 10^{-12}$	7.19	$1.60 \cdot 10^{-9}$	4.43	$3.05 \cdot 10^{-8}$	3.53
12	$5.54 \cdot 10^{-13}$	7.09	$3.65 \cdot 10^{-10}$	4.38	$8.75 \cdot 10^{-9}$	3.49
13	$8.94 \cdot 10^{-14}$	6.20	$8.42 \cdot 10^{-11}$	4.34	$2.54 \cdot 10^{-9}$	3.45
14	$2.33 \cdot 10^{-14}$	3.83	$1.96 \cdot 10^{-11}$	4.30	$7.40 \cdot 10^{-10}$	3.42
15			$4.54 \cdot 10^{-12}$	4.31	$2.18 \cdot 10^{-10}$	3.40
16			$1.08 \cdot 10^{-12}$	4.20	$6.45 \cdot 10^{-11}$	3.38
17			$2.67 \cdot 10^{-13}$	4.05	$1.92 \cdot 10^{-11}$	3.36
18			$6.11 \cdot 10^{-14}$	4.37	$5.75 \cdot 10^{-12}$	3.34
19			$1.65 \cdot 10^{-14}$	3.69	$1.73 \cdot 10^{-12}$	3.32
20			$2.53 \cdot 10^{-14}$	0.65	$5.34 \cdot 10^{-13}$	3.24

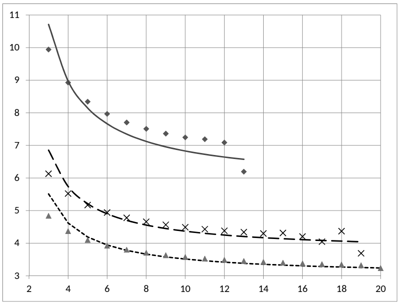

Figure 2. Ratios of the errors for $T = 1$ (W), $T = 2$ ($\times$) and $T = 3$ ($\blacktriangle$) and theoretical ratios if the convergence rate was $CN^{-3/2}r^{-N}$ with $r = 5.83$ (continuous line), $r = 3.73$ (dashed line) and $r = 3$ (dotted line).

We used $N = 2, 3, 4, \ldots, 20$ and calculated the errors in the maximum norm on $t \in [0, T]$ and the ratios of the corresponding errors. If the error behaves as Cr^{-N}, the ratios should be approximately r; if the errors behave as $CN^{-\beta}r^{-N}$, then the ratios should be $r\left(\frac{N}{N-1}\right)^{\beta}$. The results are presented in Table 2.

In Figure 2 the ratios of errors are presented for $T = 1$, $T = 2$ and $T = 3$ together with theoretical ratios if the convergence rate was $CN^{-3/2}r^{-N}$.

7 Conclusions

We derived results about the existence and uniqueness of analytic solutions for general cordial Volterra integral equations of the second kind and certain singular fractional integro-differential equations. We proposed the polynomial collocation method, where the collocation points are the Chebyshev nodes, for solving these equations and showed that the convergence of approximate solutions to the exact solution is exponential in the number of variables. Numerical examples were also given, which showed that in some cases the convergence rates may be even better than predicted.

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