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ON SOME NEW PARANORMED SEQUENCE SPACES OF FUZZY NUMBERS DEFINED BY ORLICZ FUNCTIONS AND STATISTICAL CONVERGENCE

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Abstract. In this paper we introduce the concept of strongly $\lambda(p)$ convergence of fuzzy numbers with respect to an Orlicz function and examine some properties of the resulting sequence spaces and $\lambda(p)$ – statistical convergence. It is also shown that if a sequence of fuzzy numbers is strong $\lambda(p)$ convergent with respect to an Orlicz function then it is $\lambda(p)$– statistically convergent.
Key words: Orlicz function,paranorm, de la Vallee-Poussin means, fuzzy numbers

1. Introduction and Preliminaries

The concept of paranorm is closely related to linear metric spaces.It is a generalization of an absolute value definition. Let X be a linear space. A function $g : X \to R$ is called paranorm, if

i) $g(0) = 0$,
ii) $g(x) \geq 0$, for all $x \in X$,
iii) $g(-x) = g(x)$, for all $x \in X$,
iv) $g(x + y) \leq g(x) + g(y)$, for all $x, y \in X$,
v) if (α_n) is a sequence of scalars with $\alpha_n \to \alpha$ $(n \to \infty)$ and $\{x_n\}$ a sequence of vectors with $g(x_n - x) \to 0$ $(n \to \infty)$, then $g(\alpha_n x_n - \alpha x) \to 0$ $(n \to \infty)$.

The last property is called continuity of multiplication by scalars. The space is called the paranormed space X with the paranorm g.

A function $M : [0, \infty[\to [0, \infty[$ is an Orlicz function if it is continuous, non-decreasing and convex with

$$M(0) = 0, \quad M(x) > 0 \text{ for } x > 0, \quad M(x) \to \infty \text{ as } x \to \infty.$$

An Orlicz function is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$, such that

$$M(2u) \leq KM(u), \quad u \geq 0.$$

Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to construct the sequence space

$$l_M = \Big\{x = (x_k) : \sum_k M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}.$$

The space l_M with the norm

$$\|x\| = \inf\{\rho > 0 : \sum_k M(\frac{|x_k|}{\rho}) \leq 1\}$$

becomes a Banach space which is called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x) = x^p$, $1 \leq p < \infty$.

In the later stage different Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [10], Esi, Isik and Esi [4], Esi [2], Esi and Et [3], and many others.

The purpose of this paper is to introduce and study the concepts of$\lambda(p)$–strong convergence of fuzzy numbers with respect to an Orlicz function and $\lambda(p)$–statistical convergence and some relations between them.

Let $p = (p_k) \in l_\infty$, then the following well-known inequality will be used in the paper: for sequences (a_k) and (b_k) of complex numbers we have

$$|a_k + b_k|^{p_k} \leq K(|a_k|^{p_k} + |b_k|^{p_k})$$

where $K = \max(1, 2^{H-1})$ and $H = \sup_k p_k$.

We now give here a brief introduction about the sequences of fuzzy numbers (see [1] and [12]). Let D denote the set of all bounded intervals $A = [\underline{A}, \overline{A}]$ on the real line R. For $A, B \in D$, define

$$A \leq B \text{ if and only if } \underline{A} \leq \underline{B} \text{ and } \overline{A} \leq \overline{B},$$
$$d(A, B) = \max\Big\{\Big|\underline{A} - \underline{B}\Big|, \Big|\overline{A} - \overline{B}\Big|\Big\}.$$

Then it can be easily seen that d defines a metric on D and (D, d) is a complete metric space [1]. Also the relation $\leq$ is a partial order on D.

A fuzzy number is a fuzzy subset of the real line R which is bounded, convex and normal. Let $L(R)$ denote the set of all fuzzy numbers which are upper semicontinuous and have compact support, i.e. if $X \in L(R)$ then for any $\alpha \in [0, 1]$, X^α is compact, where

$$X^\alpha = \{t : X(t) \geq \alpha, \quad \text{if } \alpha \in (0, 1]\}$$
$$X^0 = cl(\{t \in R : X(t) > \alpha, \quad \text{if } \alpha = 0\}),$$

where $cl(A)$ is the closure of A. The set R of real numbers can be embedded in $L(R)$ if we define $\bar{r} \in L(R)$ by

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r, \\ 0, & \text{if } t \neq r. \end{cases}$$

The additive identity and multiplicative identity of $L(R)$ are denoted by $\bar{0}$ and $\bar{1}$, respectively. Then the arithmetic operations on $L(R)$ are defined as follows:

$$\begin{aligned} (X \oplus Y)(t) &= \sup\{X(s) \wedge Y(t-s)\}, \quad t \in R, \\ (X \Theta Y)(t) &= \sup\{X(s) \wedge Y(s-t)\}, \quad t \in R, \\ (X \otimes Y)(t) &= \sup\{X(s) \wedge Y(t/s)\}, \quad t \in R, \\ (X/Y)(t) &= \sup\{X(st) \wedge Y(s)\}, \quad t \in R, \end{aligned}$$

These operations can be defined in terms of α–level sets as follows:

$$\begin{aligned} [X \oplus Y]^\alpha &= [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha], \\ [X \Theta Y]^\alpha &= [a_1^\alpha - b_1^\alpha, a_2^\alpha - b_2^\alpha], \\ [X \otimes Y]^\alpha &= [\min_{i \in \{1,2\}} a_i^\alpha b_i^\alpha, \max_{i \in \{1,2\}} a_i^\alpha b_i^\alpha], \\ [X^{-1}]^\alpha &= [(a_2^\alpha)^{-1}, (a_1^\alpha)^{-1}], a_i^\alpha > 0 \end{aligned}$$

for each $0 < \alpha \leq 1$.

For $r \in R$ and $X \in L(R)$, the product rX is defined as follows:

$$rX(t) = \begin{cases} X(r^{-1}t), & \text{if } r \neq 0 \\ 0, & \text{if } r = 0. \end{cases}$$

Define a map

$$\bar{d} : L(R) \times L(R) \to R_+ \cup \{0\}$$

by $\bar{d}(X,Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha)$. For $X, Y \in L(R)$ define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$ for any $\alpha \in [0,1]$. It is known that $(L(R), \bar{d})$ is a complete metric space [6].

A metric on $L(R)$ is said to be a translation invariant if

$$\bar{d}(X+Z, Y+Z) = \bar{d}(X,Y), \text{ for } X, Y, Z \in L(R).$$

Lemma 1. [7]. *If $\bar{d}$ is a translation invariant metric on $L(R)$ then*

$$i) \ \bar{d}(X+Y, \bar{0}) \leq \bar{d}(X, \bar{0}) + \bar{d}(Y, \bar{0}),$$

$$ii) \ \bar{d}(\lambda X, \bar{0}) \leq |\lambda| \bar{d}(X, \bar{0}), \quad |\lambda| > 1.$$

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set N of natural numbers into $L(R)$. The fuzzy number X_k denotes the value of the function at $k \in N$.

A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in N\}$ of fuzzy numbers is bounded.

A sequence $X = (X_k)$ of fuzzy numbers is said to be converge to a fuzzy number X_0 if for every $\varepsilon > 0$ there is a positive integer N such that $\bar{d}(X_k, X_0) < \varepsilon$ for $k > N$.

2. Some New Sequence Spaces

Recently, Nuray and Savaş [9] have defined the following space of fuzzy numbers:

$$l(p) = \{X = (X_k) : \sum_k \bar{d}(X_k, \bar{0})^{p_k} < \infty\},$$

where $p = (p_k)$ is a bounded sequence of strictly positive real numbers. If $p_k = p$ for all k, then $l(p) = l_p$, the space due to Nanda [8]. Lately, Mursaleen and Basarir [7] have defined the following spaces of sequences of fuzzy numbers:

$$F(p) = \{X = (X_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \bar{d}(X_k, X)^{p_k} = 0\},$$

$$F_0(p) = \{X = (X_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \bar{d}(X_k, \bar{0})^{p_k} = 0\},$$

$$F_\infty(p) = \{X = (X_k) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \bar{d}(X_k, \bar{0})^{p_k} < \infty\}$$

and called them the spaces of sequences of fuzzy numbers which are strongly convergent to X_0, strongly convergent to zero and strongly bounded, respectively.

In this paper, we define the following spaces:

DEFINITION 1. Let $\Lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \le \lambda_n + 1$ and let M be an Orlicz function, $p = (p_k)$ be any sequence of strictly positive real numbers and $X = (X_k)$ be sequence of fuzzy numbers, then for some ρ

$$F[M, \lambda, p] = \Big\{X = (X_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[M\Big(\frac{\bar{d}(X_k, X_0)}{\rho}\Big)\Big]^{p_k} = 0\Big\},$$

$$F_0[M, \lambda, p] = \Big\{X = (X_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[M\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big]^{p_k} = 0\Big\},$$

$$F_\infty\big[M,\lambda,p\big]=\Big\{X=\big(X_k\big):\ \sup_n\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\big)\Big]^{p_k}<\infty\Big\},$$

where $I_n=[n-\lambda_n+1,n]$.

We denote $F\big[M,\lambda,p\big]$, $F_0\big[M,\lambda,p\big]$ and $F_\infty\big[M,\lambda,p\big]$ as $F\big[M,\lambda\big]$, $F_0\big[M,\lambda\big]$ and $F_\infty\big[M,\lambda\big]$ when $p_k=1$ for all k. If $X=\big(X_k\big)\in F\big[M,\lambda,p\big]$, we say that $X=\big(X_k\big)$ is strongly $\lambda\big(p\big)$–convergent to fuzzy number X_0 with respect to the Orlicz function M. If $M\big(x\big)=x$ and $\lambda_n=n$ then

$$F\big[M,\lambda,p\big]=F\big(p\big),\quad F_0\big[M,\lambda,p\big]=F_0\big(p\big),\quad F_\infty\big[M,\lambda,p\big]=F_\infty\big(p\big),$$

which were defined by Mursaleen and Basarir [7].

3. Main Results

In this section we examine some topological properties of spaces $F\big[M,\lambda,p\big]$, $F_0\big[M,\lambda,p\big]$ and $F_\infty\big[M,\lambda,p\big]$. If d is a translation invariant, we have the following theorem.

Theorem 1. *For any Orlicz function M and any sequence $p=(p_k)$ of strictly positive real numbers, $F\big[M,\lambda,p\big]$, $F_0\big[M,\lambda,p\big]$ and $F_\infty\big[M,\lambda,p\big]$ are linear spaces over the set of complex numbers.*

Proof. We shall prove the theorem only for $F_0\big[M,\lambda,p\big]$. The other cases can be treated similarly. Let $X=\big(X_k\big)$, $Y=\big(Y_k\big)\in F_0\big[M,\lambda,p\big]$ and $\alpha,\beta\in C$. In order to prove the result we need to find some $\rho_3>0$ such that

$$\lim_n\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(\alpha X_k+\beta Y_k,\bar{0})}{\rho_3}\Big)\Big]^{p_k}=0.$$

Since $X=\big(X_k\big)$, $Y=\big(Y_k\big)\in F_0\big[M,\lambda,p\big]$, there exist some $\rho_1>0$ and $\rho_2>0$ such that

$$\lim_n\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho_1}\Big)\Big]^{p_k}=0,\quad \lim_n\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(Y_k,\bar{0})}{\rho_2}\Big)\Big]^{p_k}=0.$$

Define $\rho_3=\max\big(2\big|\alpha\big|\rho_1,2\big|\beta\big|\rho_2\big)$. Since M is non decreasing and convex, we get

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(\alpha X_k+\beta Y_k,\bar{0})}{\rho_3}\Big)\Big]^{p_k}\le\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(\alpha X_k,\bar{0})}{\rho_3}+\frac{\bar{d}(\beta Y_k,\bar{0})}{\rho_3}\Big)\Big]^{p_k}$$

$$\leq \frac{1}{\lambda_n}\sum_{k\in I_n}\frac{1}{2^{p_k}}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho_1}\Big)+M\Big(\frac{\bar{d}(Y_k,\bar{0})}{\rho 2}\Big)\Big]^{p_k}$$

$$\leq \frac{1}{\lambda_n}\sum_{k\in I_n}\frac{1}{2^{p_k}}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho_1}\Big)+M\Big(\frac{\bar{d}(Y_k,\bar{0})}{\rho 2}\Big)\Big]^{p_k}$$

$$\leq \frac{K}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho_1}\Big)\Big]^{p_k}+\frac{K}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(Y_k,\bar{0}\Big)}{\rho_2}\Big)\Big]^{p_k}\to 0 \text{ as } n\to\infty,$$

where $K=\max\big(1,2^{H-1}\big)$, $H=\sup_k p_k$, so that $\alpha X+\beta Y\in F_0\big[M,\lambda,p\big]$. This completes the proof. ■

Theorem 2. *For any Orlicz function M and a bounded sequence $p=(p_k)$ of strictly positive real numbers, $F\big[M,\lambda,p\big]$ and $F_0\big[M,\lambda,p\big]$ are paranormed spaces with*

$$g(X)=\inf\Big\{\rho^{p_n/H}:\ \Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho_1}\Big)\Big]^{p_k}\Big)^{1/M}\leq 1,\ \ n\geq 1\Big\},$$

where $M=\max\big(1,H\big)$.

Proof. Clearly $g(\bar{0})=0$ and $g(X)=g(-X)$. Since $\bar{d}$ is a translation invariant, it can be seen easily that

$$g(X+Y)\leq g(X)+g(Y),\text{ for } X=(X_k),\ Y=(Y_k)\in F_0\big[M,\lambda,p\big].$$

Since $M(0)=0$, we get $\inf\big\{\rho^{p_n/H}\big\}=0$ for $X=\bar{0}$. Conversely, suppose that $g(X)=0$, then

$$\inf\Big\{\rho^{p_n/H}:\Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)\Big]^{p_k}\Big)^{1/M}\leq 1,\ \ n\geq 1\Big\}=0.$$

This implies that for a given $\varepsilon>0$, there exists some $\rho_\varepsilon\big(0<\rho_\varepsilon<\varepsilon\big)$ such that

$$\Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho_\varepsilon}\Big)\Big]^{p_k}\Big)^{1/M}\leq 1.$$

Thus for each n we get

$$\Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\varepsilon}\Big)\Big]^{p_k}\Big)^{1/M}\leq\Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho_\varepsilon}\Big)\Big]^{p_k}\Big)^{1/M}\leq 1.$$

Suppose that $\bar{d}\big(X_{n_m},0\big)\neq 0$ for some $m\in I_n$. Let $\varepsilon\to 0$, then $\big(\frac{\bar{d}(X_{n_m},\bar{0})}{\varepsilon}\big)\to\infty$. It follows that

$$\big(\frac{1}{\lambda_n}\sum_{k\in I_n}\big[M\big(\frac{\bar{d}(X_{n_m},\bar{0})}{\varepsilon}\big)\big]^{p_k}\big)^{1/M}\to\infty,$$

which is a contradiction. Therefore $X_{n_m} \neq 0$. Finally, we prove that scalar multiplication is continuous. Let γ be any complex number. By definition

$$g(\gamma X) = \inf\Big\{\rho^{p_n/H} : \Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(\gamma X_k,\bar{0})}{\rho}\Big)\Big]^{p_k}\Big)^{1/M} \leq 1, \ \ n \geq 1\Big\}.$$

Then

$$g(\gamma X) = \inf\Big\{\big(|\gamma|t\big)^{p_n/H}\colon \Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{t}\Big)\Big]^{p_k}\Big)^{1/M} \leq 1, \ \ n \geq 1\Big\},$$

where $t = \dfrac{\rho}{|\gamma|}$. Since $|\gamma|^{p_k} \leq \max\big(1, |\gamma|^H\big)$, we have

$$g(\gamma X) \leq \big(\max\big(1, |\gamma|^H\big)\big)^{1/M} \inf\Big\{t^{p_n/H} :$$
$$\Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{t}\Big)\Big]^{p_k}\Big)^{1/M} \leq 1, \ \ n \geq 1\Big\}.$$

So, the fact that a scalar multiplication is continuous follows from the above inequality. ■

Theorem 3. *Let* $0 < h = \inf_k p_k \leq \sup_k p_k = H < \infty$. *For any Orlicz function* M *which satisfies* Δ_2*–condition, we have* $F[\lambda, p] \subset F[M, \lambda, p]$, *where*

$$F\big[\lambda, p\big] = \Big\{X = (X_k) : \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(X_k, X_0)}{\rho}\Big)\Big]^{p_k} = 0\Big\}$$

for some $\rho > 0$.

Proof. Let $X = \big(X_k\big) \in F\big[\lambda, p\big]$ so that

$$\lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{\bar{d}(X_k, X_0)}{\rho}\Big)\Big]^{p_k} = 0, \text{ for some } \rho > 0.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t \leq \delta$. Denote $y_k = \dfrac{\bar{d}(X_k, X_0)}{\rho}$ and consider

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\big(y_k\big)\Big]^{p_k} < \lambda_n \max\big(\varepsilon, \varepsilon^h\big)$$

by using continuity of M. For the second summation, we will make the following procedure. We have

$$y_k \leq \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}.$$

Since M is non-decreasing and convex, it follows that

$$M(y_k) < M\Big(1+\frac{y_k}{\delta}\Big) \le \frac{1}{2}M(2) + \frac{1}{2}M\Big(\frac{2y_k}{\delta}\Big).$$

Since M satisfies Δ_2–condition, we can write

$$M(y_k) \le \frac{L}{2}\,\frac{y_k}{\delta}M(2) + \frac{L}{2}\,\frac{y_k}{\delta}M(2) = L\,\frac{y_k}{\delta}M(2).$$

We get the following estimates

$$\sum_{\substack{k\in I_n\\ y_k>\delta}} \big[M(y_k)\big]^{p_k} \le \max\Big(1,\big[LM(2)\,\delta^{-1}\big]^H\Big)\lambda_n\frac{1}{\lambda_n}\sum_{k\in I_n}\big[(y_k)\big]^{p_k},$$

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\big[M(y_k)\big]^{p_k} \le \max\big(\varepsilon,\varepsilon^h\big) + \max\big(1,\big[\frac{L}{\delta}M(2)\big]^H\big)\frac{1}{\lambda_n}\sum_{k\in I_n}\big[(y_k)\big]^{p_k}.$$

Taking the limits $\varepsilon\to 0$ and $n\to\infty$, it follows that $X=\big(X_k\big)\in F\big[M,\lambda,p\big]$.
■

Theorem 4. *Let* $0\le p_k\le q_k$ *and* $\big(\frac{q_k}{p_k}\big)$ *be bounded. Then*

$$F\big[M,\lambda,q\big] \subset F\big[M,\lambda,p\big].$$

Proof. The theorem is proved by using the same technique as in the proof of Theorem 3.3 by Murseleen and Basarir [7]. ■

Now, we give some well-known definitions:

DEFINITION 2. A sequence $X=\big(X_k\big)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number X_0 if for every $\varepsilon>0$,

$$\lim_n \frac{1}{n}\big|\big\{k\le n:\bar{d}\big(X_k,X_0\big)\ge\varepsilon\big\}\big| = 0.$$

We note that if a sequence $X=\big(X_k\big)$ of fuzzy numbers converges to a fuzzy number X_0, then it statistically converges to X_0. But the converse statement is not necessarily valid.

DEFINITION 3. A sequence $X=\big(X_k\big)$ of fuzzy numbers is said to be $\lambda(p)$–statistically convergent or $S_{\lambda(p)}$ convergent to a fuzzy number X_0 if for every $\varepsilon>0$

$$\lim_n \frac{1}{n}\Big|\Big\{k\in I_n:\ \big[\bar{d}\big(X_k,X_0\big)\big]^{p_k}\ge\varepsilon\Big\}\Big| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case, we write

$$S_{\lambda(p)} = \Big\{X=\big(X_k\big):\ \lim_n\frac{1}{\lambda_n}\Big|\Big\{k\in I_n:\ \big[\bar{d}\big(X_k,X_0\big)\big]^{p_k}\ge\varepsilon\Big\}\Big| = 0\Big\}.$$

In the case $p_k=1$ for all k, we obtain λ–statistically convergent sequence spaces S_λ, which was defined and studied by Savas [11].

Theorem 5. *The following statements are valid:*

a) $F[\lambda, p] \subset S_{\lambda(p)}$,

b) if $X = (X_k) \in l_\infty(p) \cap S_{\lambda(p)}$, *then* $X = (X_k) \in F[\lambda, p]$,

c) $l_\infty(p) \cap S_{\lambda(p)} = l_\infty(p) \cap F[\lambda, p]$,

where $l_\infty(p) = \{X = (X_k) : \sup_k [\bar{d}(X_k, X_0)]^{p_k} \le K,\ K > 0\}$.

Proof.
a) Let $\varepsilon > 0$ and $X = (X_k) \in F[\lambda, p]$. Then we have

$$\sum_{k \in I_n} [\bar{d}(X_k, X_0)]^{p_k} \ge \varepsilon^H \big|\{k \in I_n :\ [\bar{d}(X_k, X_0)]^{p_k} \ge \varepsilon\}\big|.$$

Hence $X = (X_k) \in S_{\lambda(p)}$.

b) Suppose that $X = (X_k) \in S_{\lambda(p)}$ and $X = (X_k) \in l_\infty(p)$. Since $X = (X_k)$ is bounded, we write $[\bar{d}(X_k, X_0)]^{p_k} \le T$ for all k. Given $\varepsilon > 0$, let

$$A_n = \big|\{k \in I_n :\ [\bar{d}(X_k, X_0)]^{p_k} \ge \varepsilon\}\big|,$$
$$B_n = \big|\{k \in I_n :\ [\bar{d}(X_k, X_0)]^{p_k} < \varepsilon\}\big|.$$

Then we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} [\bar{d}(X_k, X_0)]^{p_k} = \frac{1}{\lambda_n} \sum_{k \in A_n} [\bar{d}(X_k, X_0)]^{p_k} + \frac{1}{\lambda_n} \sum_{k \in B_n} [\bar{d}(X_k, X_0)]^{p_k}$$
$$\le \frac{T}{\lambda_n} |A_n| + \varepsilon^H.$$

Hence $X = (X_k) \in F[\lambda, p]$.

c) This proof follows from (a) and (b). ■

Theorem 6. *If* $\liminf_n \frac{\lambda_n}{n} > 0$, *then* $S_{(p)} \subset S_{\lambda(p)}$, *where*

$$S_{(p)} = \{X = (X_k) :\ \lim_n \frac{1}{n} \big|\{k \in I_n :\ [\bar{d}(X_k, X_0)]^{p_k} \ge \varepsilon\}\big| = 0\}.$$

Proof. Let $X = (X_k) \in S_{(p)}$. For given $\varepsilon > 0$, we get

$$\{k \le n :\ [\bar{d}(X_k, X_0)]^{p_k} \ge \varepsilon\} \supset A_n,$$

where A_n is the same as in Theorem 5. Thus,

$$\frac{1}{n} \big|\{k \le I_n :\ [\bar{d}(X_k, X_0)]^{p_k} \ge \varepsilon\}\big| \ge \frac{1}{n} |A_n| = \frac{\lambda_n}{n} \frac{1}{\lambda_n} |A_n|.$$

Taking limit as $n \to \infty$ and using $\liminf_n \frac{\lambda_n}{n} > 0$, we get $X = (X_k) \in S_{\lambda(p)}$.
■

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