Mathematical Modelling and Analysis
Volume 6 Number 1, 2001, pages 106–116

PROBLEMS ON CONJUGATION OF POLYTYPIC EQUATIONS

V.I. KORZYUK[1], S.V. LEMESHEVSKY[2]

[1] *Belarusian State University*
Skoryna Ave. 4, 220050, Minsk, Belarus
E-mail: Korzyuk@org.bsu.unibel.by

[2] *Institute of Mathematics of NAS of Belarus*
Surganov St., 11, 220072, Minsk, Belarus
E-mail: svl@im.bas-net.by

Received September 1, 2000

ABSTRACT

Some conjugation problems of hyperbolic and parabolic equations with different consistency conditions on the interface are considered. Issues concerning one-valued solvability of these problems are considered. Difference schemes for numerical solution of mentioned conjugation problems are proposed. Estimates of accuracy of algorithms suggested are obtained.

INTRODUCTION

The conjugation problems of two or more differential equations, which are given in different space domains and connected by some consistency conditions on the interface, arise when we are studying the effects in the media with different physical properties. For example, we have the conjugation problems of second-order equations of the same type in the study of a stationary and time-dependent temperature distributions of a body that consists of heterogeneous pieces and in the study of many diffraction problems, etc.

The investigation of the hydrodynamical pressure of weakly compressible fluid in the channel surrounded by porous medium, the study of the electromagnetic field and magnetic fluid dynamics effects lead to the conjugation problems of polytypic equations [12; 13]. Motion of the viscoelastic and viscous fluids in the plane horizontal split leaving out of account the surface phenomena describes by one-dimensional hyperbolic equation and heat equation supplemented with integro-differential conditions on the interface of moving fluids [1]. In this case the equation type is defined by the medium properties and character of the process. The consistency conditions on the interface of subdomains attract particular attention.

Along with numerical solution it is also actual to examine the validity statement of mathematical problem, i.e., the proof of its solvability and solution uniqueness.

We shall consider some conjugation problems of hyperbolic and parabolic equations with different consistency conditions on the interface, and present some results regarding one-valued solvability of such problems. We also study questions of numerical solving such problems.

1. CONJUGATION PROBLEM IN DOMAINS WITH MOVING BOUNDARIES

1.1. Statement of the problem

Let $Q = \{\, (t,\boldsymbol{x}) :\ c_0 t < x_1 < l_1 + c_0 t,\ 0 < x_2 < l_2,\ 0 < t < T \,\}$ be a bounded domain in the three-dimensional Euclidean space $\mathbb{R}^3$ of variables $(t,\boldsymbol{x}) = (t, x_1, x_2)$. Suppose Q is separated by the surface $\Gamma = \big\{(t,\boldsymbol{x}) :\ x_1 = \xi + c_0 t,\ 0 < \xi < l_1,\ 0 < x_2 < l_2,\ 0 < t < T\big\}$ into two subdomains, Q_1 and Q_2: $Q_1 = \{(t,\boldsymbol{x}) :\ c_0 t < x_1 < \xi + c_0 t,\ 0 < x_2 < l_2, 0 < t < T\}$, $Q_2 = \{(t,\boldsymbol{x}) :\ \xi + c_0 t < x_1 < l_1 + c_0 t,\ 0 < x_2 < l_2, 0 < t < T\}$. The boundary ∂Q of Q consists of a lower base, $\Omega^0 = \{(t,\boldsymbol{x}) \in \partial Q :\ t = 0\}$, an upper base, $\Omega^T = \{(t,\boldsymbol{x}) \in \partial Q :\ t = T\}$, and a side surface, $S = \{(t,\boldsymbol{x}) \in \partial Q :\ 0 < t < T\}$. The lower base $\overline{\Omega}^0$ consists of two parts: $\overline{\Omega}_1^0 = \overline{\Omega}^0 \cap \partial Q_1$ and $\overline{\Omega}_2^0 = \overline{\Omega}^0 \cap \partial Q_2$ ($\overline{\Omega}^0$ and $\overline{\Omega}_i^0$ are closures of Ω^0 and Ω_i^0, $i = 1, 2$, respectively).

In Q_1 we shall consider equation of hyperbolic type with respect to desired function $u^{(1)}(t,\boldsymbol{x})$

$$\frac{\partial^2 u^{(1)}}{\partial t^2} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k_i^{(1)}(\boldsymbol{x}) \frac{\partial u^{(1)}}{\partial x_i}\right) + f^{(1)}(t,\boldsymbol{x}), \quad (t,\boldsymbol{x}) \in Q_1, \tag{1.1}$$

and in Q_2 we shall consider parabolic equation with respect to function $u^{(2)}(t,\boldsymbol{x})$

$$\frac{\partial u^{(2)}}{\partial t} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k_i^{(2)}(\boldsymbol{x}) \frac{\partial u^{(2)}}{\partial x_i}\right) + f^{(2)}(t,\boldsymbol{x}), \quad (t,\boldsymbol{x}) \in Q_2, \tag{1.2}$$

where $k_i^{(m)}(\boldsymbol{x}) \in C^1(\overline{Q}_m)$ ð $0 < c_1 \leqslant k_i^{(m)}(\boldsymbol{x}) \leqslant c_2,\ i = 1, 2,\ m = 1, 2.$

In addition, assume that the coefficients of the equation (1.1) satisfy the following condition

$$k_1^{(1)}(\boldsymbol{x}) - c_0^2 \geq \delta > 0. \tag{1.3}$$

The equations (1.1) and (1.2) are supplemented with the following boundary and initial conditions:

$$u\big|_S = 0, \quad (t, \boldsymbol{x}) \in S, \tag{1.4}$$

$$u\big|_{\Omega^0} = u_0(\boldsymbol{x}), \qquad \frac{\partial u^{(1)}}{\partial t}\bigg|_{\Omega_1^0} = u_1^{(1)}(\boldsymbol{x}), \tag{1.5}$$

where

$$u(t, \boldsymbol{x}) = \begin{cases} u^{(1)}(t, \boldsymbol{x}), & (t, \boldsymbol{x}) \in \overline{Q}_1, \\ u^{(2)}(t, \boldsymbol{x}), & (t, \boldsymbol{x}) \in \overline{Q}_2, \end{cases} \quad u_0(\boldsymbol{x}) = \begin{cases} u_0^{(1)}(\boldsymbol{x}), & (0, \boldsymbol{x}) \in \overline{\Omega}_1^0, \\ u_0^{(2)}(\boldsymbol{x}), & (0, \boldsymbol{x}) \in \overline{\Omega}_2^0. \end{cases}$$

At the interface Γ the following consistency conditions are valid

$$u^{(1)}\big|_\Gamma = u^{(2)}\big|_\Gamma, \quad \left(c_0 \frac{\partial u^{(1)}}{\partial t} + k_1^{(1)}(\boldsymbol{x}) \frac{\partial u^{(1)}}{\partial x_1}\right)\bigg|_\Gamma = \left(k_1^{(2)}(\boldsymbol{x}) \frac{\partial u^{(2)}}{\partial x_1}\right)\bigg|_\Gamma. \tag{1.6}$$

1.2. Existence and uniqueness of a strong solution

In [9] the existence of a unique strong solution of the problem (1.1)–(1.6) are proven using the method of energy inequalities and mollifiers with variable step [2; 5]. In addition, the issues regarding numerical solution of this problem are considered. Below we shall adduce these results.

Let $\mathcal{B}$ be a Banach space that is a closure of a set $\{u : u^{(m)} \in C^2(\overline{Q}_m)\ (m = 1, 2),\ u$ satisfies the conditions (1.4) and (1.6)$\}$ with respect to the norm

$$\begin{aligned} \|u\|_{\mathcal{B}} = \; & \left\|\frac{\partial u^{(1)}}{\partial t}\right\|_{L_2(Q_1)} + \sup_{0 \leqslant t \leqslant T} \left(\left\|\frac{\partial u^{(2)}}{\partial t}\right\|_{L_2(\Omega^{(1)}(t))} \right. \\ & \left. + \sum_{m=1}^{2} \sum_{i=1}^{2} \left\|\frac{\partial u^{(m)}}{\partial x_i}\right\|_{L_2(\Omega^{(m)}(t))} \right), \end{aligned}$$

where $\Omega^{(m)}(t)$ is a section of the subdomain Q_m $(m = 1, 2)$ by the plane $\{(t, \boldsymbol{x}) \in \mathcal{R}^3 : \ t = \text{const}\}$, $\|\cdot\|_{L_2}$ is a norm in a space, L_2, of Lebesgue integrable functions whose squares are also Lebesgue integrable. Let $\overset{\circ}{H}{}^1(\Omega^0)$ be a Hilbert space that consists of functions $u \in L_2(\Omega^0)$ $(u = 0$ on $\overline{\Omega}^0 \cap \overline{S})$ whose the first order weak derivatives are also elements of $L_2(\Omega^0)$. The norm in $\overset{\circ}{H}{}^1(\Omega^0)$ is $\|\cdot\|_{\overset{\circ}{H}{}^1(\Omega^0)} = \|\cdot\|_{L_2(\Omega^0)} + \sum\limits_{i=1}^{2} \|\frac{\partial \cdot}{\partial x_i}\|_{L_2(\Omega^0)}$.

Denote by $\mathcal{L}$ the differential operator $\mathcal{L}u = \left(\mathcal{L}^{(1)}u^{(1)}, \mathcal{L}^{(2)}u^{(2)}\right)$, where $\mathcal{L}^{(m)}\cdot = \frac{\partial^{3-m}\cdot}{\partial t^{3-m}} - \sum\limits_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k_i^{(m)} \frac{\partial}{\partial x_i}\right)$, $m = 1, 2$. We can consider the problem (1.1), (1.2), (1.4)—(1.6) as the following operator equation

$$Lu = F, \quad Lu = \left(\mathcal{L}u, l_0 u, l_1 u^{(1)}\right), \quad l_0 u = u\big|_{\Omega^0}, \quad l_1 u^{(1)} = \frac{\partial u^{(1)}}{\partial t}\bigg|_{\Omega_1^0},$$

$$F = \left(f(t,\boldsymbol{x}), u_0(\boldsymbol{x}), u_1^{(1)}(\boldsymbol{x})\right), \quad f(t,\boldsymbol{x}) = \begin{cases} f^{(1)}(t,\boldsymbol{x}), & (t,\boldsymbol{x}) \in Q_1, \\ f^{(2)}(t,\boldsymbol{x}), & (t,\boldsymbol{x}) \in Q_2, \end{cases}$$

acting from $\mathcal{B}$ onto $\mathcal{H} = L_2(Q) \times \overset{\circ}{H}{}^1(\Omega^0) \times L_2(\Omega_1^0)$ and which domain of definition is $\mathcal{D}(L) = \{u(t,\boldsymbol{x}) : \ u^{(m)}(t,\boldsymbol{x}) \in C^2(\overline{Q}_m), \ m = 1,2, \ u(t,\boldsymbol{x})$ satisfies the conditions (1.4) and (1.6)$\}$.

For the differential problem (1.1), (1.2), (1.4)–(1.6) the following statement is valid.

Theorem 1.1 [9]. *Suppose that* $k_i^{(m)}(\boldsymbol{x}) \in C^1(\overline{Q}_m)$ ð $0 < c_1 \leqslant k_i^{(m)}(\boldsymbol{x}) \leqslant c_2, \ i = 1,2, \ m = 1,2$ *and assume that the condition* (1.3) *holds; then for the conjugation problem* (1.1), (1.2) (1.4) – (1.6) *the following estimate is valid*

$$\|u\|_{\mathcal{B}} \leqslant c\,\|Lu\|_{\mathcal{H}} = c\left(\|\mathcal{L}u\|_{L_2(Q)} + \|l_0 u\|_{\overset{\circ}{H}{}^1(\Omega^0)} + \left\|l_1 u^{(1)}\right\|_{L_2(\Omega_1^0)}\right), \quad c > 0. \tag{1.7}$$

Operator $L : \mathcal{B} \to \mathcal{H}$ admits a closure $\overline{L}$ [5]. The solution of the operator equation $\overline{L}u = F$ is a *strong solution* of the problem (1.1), (1.2), (1.4)—(1.6).

Theorem 1.2 [9]. *Under the conditions of Theorem 1.1 for arbitrary* $F \in \mathcal{H}$ *there exists a unique strong solution* $u \in \mathcal{B}$ *of the problem* (1.1), (1.2), (1.4) – (1.6). *In addition,*

$$\|u\|_{\mathcal{B}} \leqslant c\|F\|_{\mathcal{H}}, \qquad c > 0. \tag{1.8}$$

1.3. Difference scheme

Let $\omega_{h\tau} = \omega_h \times \omega_\tau$ be a uniform moving mesh in the domain Q. Here

$$\overline{\omega}_h = \Big\{\boldsymbol{x}_{i_1 i_2}^j = \left(x_{1i_1}^j, x_{2i_2}\right) : \ x_{1i_1}^j = i_1 h_1 + c_0 t_j; \quad x_{2i_2} = i_2 h_2, \\ 0 \leqslant i_k \leqslant N_k, \quad h_k N_k = l_k, \quad k = 1,2\Big\},$$

$$\omega_\tau = \{t_j = j\tau : \quad 0 < j \leqslant N_0 - 1, \quad \tau N_0 = T\}.$$

The set $\omega_h = \Big\{\boldsymbol{x}_{i_1 i_2}^j : \quad 0 < i_k < N_k, \quad k = 1,2\Big\}$ is a set of interior mesh-points of $\overline{\omega}_h$, and $\partial\omega_h = \overline{\omega}_h \backslash \omega_h = \{\boldsymbol{x}_{i_1 i_2}^j : \ i_1 = 0, N_1, \ 0 < i_2 < N_2$ and $0 < i_1 < N_1, \ \ i_2 = 0, N_2\}$ is a set of boundary mesh-points of $\overline{\omega}_h$.

We assume that the interface Γ contains the mesh-points of $\omega_{h\tau}$, and denote this set by $\gamma_h = \Big\{\boldsymbol{x}^j_{p_1 i_2} = \Big(x^j_{p_1}, x^j_{i_2}\Big) : \ x^j_{1p_1} = p_1 h_1 + c_0 t_j, \ p_1 h_1 = \xi,$
$0 < i_2 < N_2\Big\}$, where $2 \leqslant p_1 \leqslant N_1 - 2$. In addition, in the domains Q_1 and Q_2 we shall consider the following meshes $\omega_1 = \omega_{1h} \times \omega_\tau, \ \omega_2 = \omega_{2h} \times \omega_\tau$. Here

$$\omega_{1h} = \Big\{\boldsymbol{x}^j_{i_1 i_2} : \ 0 < i_1 < p_1, \ 0 < i_2 < N_2\Big\},$$
$$\omega_{2h} = \Big\{\boldsymbol{x}^j_{i_1 i_2} : \ p_1 < i_1 < N_1, \ 0 < i_2 < N_2 - 1\Big\}.$$

On the moving mesh $\omega_{h\tau}$ we approximate the differential problem (1.1), (1.2), (1.4) – (1.6) by a three-layered scheme

$$y_{\bar{t}t} = (a_1 y_{\bar{x}_1})^{(\sigma_1,\sigma_2)}_{x_1} + (a_2 y_{\bar{x}_2})^{(\sigma_1,\sigma_2)}_{x_2} + 2c_0 y_{t\overset{\circ}{x}_1} + \varphi, \quad (t,\boldsymbol{x}) \in \omega_1, \tag{1.9}$$

$$y_t = (a_1 y_{\bar{x}_1})^{(\sigma_1,\sigma_2)}_{x_1} + (a_2 y_{\bar{x}_2})^{(\sigma_1,\sigma_2)}_{x_2} + c_0 y_{\overset{\circ}{x}_1} + \varphi, \quad (t,\boldsymbol{x}) \in \omega_2, \tag{1.10}$$

$$y\big|_{\partial\omega_h} = 0, \quad (t,\boldsymbol{x}) \in \partial\omega_h, \tag{1.11}$$

$$y(0,\boldsymbol{x}) = u_0(\boldsymbol{x}), \quad y_t(0,\boldsymbol{x}) = u_1^{(1)}(\boldsymbol{x}), \quad \boldsymbol{x} \in \omega^+_{1h}, \qquad \omega^+_{1h} = \omega_{1h} \cup \gamma_h, \tag{1.12}$$

$$y(0,\boldsymbol{x}) = u_0(\boldsymbol{x}), \quad y_t(0,\boldsymbol{x}) = u_1^{(2)}(\boldsymbol{x}), \quad \boldsymbol{x} \in \omega_{2h}, \tag{1.13}$$

with constant weights $\sigma_k, \quad k = 1,2$. Here $u_1^{(2)}(\boldsymbol{x}) = \mathrm{L}u_0(\boldsymbol{x}) + c_0 \frac{\partial u_0(\boldsymbol{x})}{\partial x_1} + f^{(1)}(0,\boldsymbol{x}), \ \boldsymbol{x} \in \omega_{2h}, \mathrm{L}u = \frac{\partial}{\partial x_1}\left(k_1^{(1)} \frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k_2^{(1)} \frac{\partial u}{\partial x_2}\right)$. We use the standard notation of the theory of difference schemes [8]:

Stencil functionals φ and $a_m(\boldsymbol{x})$ $(m = 1,2)$ are defined by formulas

$$\varphi(\boldsymbol{x}) = 0{,}5\Big(f(t, x_1 - 0{,}5h_1, x_2) + f(t, x_1 - 0{,}5h_1, x_2)\Big),$$

$$a_1(\boldsymbol{x}) = \begin{cases} k_1^{(1)}(x_1 - 0{,}5h_1, x_2) - c_0^2, & \boldsymbol{x} \in \omega^+_{1h}, \\ k_1^{(2)}(x_1 - 0{,}5h_1, x_2), & \boldsymbol{x} \in \omega_{2h}, \end{cases}$$

$$a_2(\boldsymbol{x}) = \begin{cases} k_2^{(1)}(x_1, x_2 - 0{,}5h_2), & \boldsymbol{x} \in \omega^+_{1h}, \\ k_2^{(2)}(x_1, x_2 - 0{,}5h_2), & \boldsymbol{x} \in \omega_{2h}, \end{cases}$$

respectively.

Similarly to [8], we approximate the second consistency conditions (1.6) with the second order of accuracy with respect to spatial variables and write its approximation in the following form

$$\frac{c_0}{h_1} y_t + 0{,}5\,(y_t + y_{\bar{t}t}) = (a_1 y_{\bar{x}_1})^{(\sigma_1,\sigma_2)}_{x_1} + (a_2 y_{\bar{x}_2})^{(\sigma_1,\sigma_2)}_{x_2} + c_0 y_{t\bar{x}_1} + 0{,}5 c_0 y_{x_1} + \varphi. \tag{1.14}$$

Note that the second initial condition $y_t(0,\boldsymbol{x}) = u_1^{(2)}(\boldsymbol{x})$ for the hyperbolic equation is obtained from the condition of the second order of accuracy of value $y(\tau,\boldsymbol{x})$ [8].

The following statement holds.

Theorem 1.3 [9]. *Suppose that the conditions* (1.3) *and* $k_i^{(m)} \in C^3(Q_m) \cap C^2(\Gamma)$ $(i = 1,2)$, $u^{(m)} \in C^4(Q_m) \cap C^3(\Gamma)$, $m = 1,2$, $k_2^{(1)}(\boldsymbol{x}) = k_2^{(2)}(\boldsymbol{x})$ *for* $\boldsymbol{x} \in \Gamma$ *are valid. Then under conditions* $\sigma_1 \geq \sigma_2 + 0{,}5$, $\sigma_2 > 0$, *the solution of the difference scheme* (1.9) – (1.14) *converges to the solution of the differential problem* (1.1), (1.2), (1.4) – (1.6), *and for the error* $z(t) = y(t) - u(t)$ *the following estimate holds*

$$\max_{t\in\omega_\tau} \|z(t)\|_{A_0} \leqslant M_1(\tau + h_1^{3/2} + h_2^2),$$

where $M_1 > 0$ *and* $A_0 y = -\sum\limits_{k=1}^{2} (a_k(\boldsymbol{x}) y_{\bar{x}_k})_{x_k}$ *for* $\boldsymbol{x} \in \omega_h$, $A_0 y = 0$ *for* $\boldsymbol{x} \in \partial\omega_h$.

2. CONJUGATION PROBLEM WITH INTEGRO-DIFFERENTIAL CONDITIONS ON THE SUB-DOMAINS INTERFACE

We have noted above that the problem on conjugation of equations of hyperbolic and parabolic types with integro-differential conditions on the sub-domains interface arise in description of some physical phenomena. In [6; 7] the questions concerning one-valued solvability and numerical solution of the problem mentioned are considered.

The motion of the viscoelastic and viscous fluids in the plane horizontal split leaving out of account the surface phenomena describes by the following one-dimensional problem

$$\mathcal{L}^{(1)} u^{(1)} \equiv \theta \frac{\partial^2 u^{(1)}}{\partial t^2} + \frac{\partial u^{(1)}}{\partial t} - \frac{\partial^2 u^{(1)}}{\partial x^2} = f^{(1)}(t,x), \quad (t,x) \in Q^{(1)} \tag{2.1}$$

$$\mathcal{L}^{(2)} u^{(2)} \equiv \frac{1}{\rho}\frac{\partial u^{(2)}}{\partial t} - \frac{1}{\mu}\frac{\partial^2 u^{(2)}}{\partial x^2} = f^{(2)}(t,x), \quad (t,x) \in Q^{(2)} \tag{2.2}$$

$$\ell u \equiv u(0,x) = u_0(x), \quad x \in (0,l), \tag{2.3}$$

$$\ell_1 u \equiv \frac{\partial u^{(1)}(0,x)}{\partial t} = u_1^{(1)}(x), \quad x \in (0,\xi), \tag{2.4}$$

$$u(t,0) = u(t,l) = 0, \quad t \in (0,T), \tag{2.5}$$

$$u^{(1)}\big|_\gamma = u^{(2)}\big|_\gamma, \tag{2.6}$$

$$\frac{1}{\theta}\int\limits_0^t \left(\exp\left(\frac{t'-t}{\theta}\right)\frac{\partial u^{(1)}}{\partial x}\right)\Big|_\gamma dt' = \frac{1}{\mu}\left.\frac{\partial u^{(2)}}{\partial x}\right|_\gamma . \tag{2.7}$$

2.1. Difference scheme

Further we shall assume that the following conditions hold

$$f^{(i)}(t,x)\in C\left(Q^{(i)}\right),\ u^{(i)}(t,x)\in C^4\left(Q^{(i)}\right),\ i=1,2,\ u_0(x)\in C^2(0,l). \tag{2.8}$$

On the interval $[0,T]$ let us introduce the uniform grid $\omega_\tau=\{t_j=j\tau,\ j=1,2,\dots N_t-1,\ N_t\tau=T\}$. In the domains $Q^{(1)}$ and $Q^{(2)}$ we shall consider the uniform grids $\omega_1=\omega_{1h_1}\times\omega_\tau$ and $\omega_2=\omega_{2h_2}\times\omega_\tau$, respectively. Here $\overline{\omega}_{1h_1}=\{x_i=ih_1,\ i=0,1,2,\dots N_1,\ N_1h_1=\xi\}$, $\overline{\omega}_{2h_2}=\{x_{p+i}=\xi+ih_1,\ i=0,1,2,\dots N_2,\ N_2h_2=l-\xi\}$. Let $\overline{\omega}_h=\overline{\omega}_{1h_1}\cup\overline{\omega}_{2h_2}$, $\overline{\omega}=\overline{\omega}_h\times\omega_\tau$.

We approximate the problem (2.1) — (2.7) on the grid ω by the following implicit difference scheme

$$\theta y_{1\bar{t}t}+\frac{\theta}{\tau}\left(\exp\left(\frac{\tau}{\theta}\right)-1\right)y_{1t}=\widehat{y}_{1\bar{x}x}+\varphi_1,\quad (t,x)\in\omega_1, \tag{2.9}$$

$$\frac{1}{\rho}y_{2t}=\frac{1}{\mu}\widehat{y}_{2\bar{x}x}+\varphi_2,\quad (t,x)\in\omega_2, \tag{2.10}$$

$$y_1(t,0)=y_2(t,l)=0,\quad t\in\omega_\tau, \tag{2.11}$$

$$y(0,x)=u_0(x),\qquad y_t(0,x)=\overline{u}_1(x),\quad x\in\omega_h, \tag{2.12}$$

$$\begin{gathered} y_1(t,\xi)=y_2(t,\xi),\quad t\in\omega_\tau \\ \frac{1}{\theta}\widehat{J}y_{1\bar{x}}+0{,}5h_1\left(y_{1t}-\exp\left(-\frac{\tau}{\theta}\right)y_{1t}(0)-\varphi_1\right)= \\ =\frac{1}{\mu}\widehat{y}_{2x}-0{,}5h_2\left(y_{2t}-\varphi_2\right),\quad x=\xi,\quad t\in\omega_\tau, \end{gathered} \tag{2.13}$$

where

$$\overline{u}_1(x)=\begin{cases}\left(1-\dfrac{\tau}{2\theta}\right)u_1^{(1)}(x)+\dfrac{\tau}{2\theta}\left(\dfrac{\partial^2u_0(x)}{\partial x^2}+f^{(1)}(0,x)\right), & x\in\omega_{1h_1}\cup\{\xi\},\\ \rho\left(\dfrac{1}{\mu}\dfrac{\partial^2u_0(x)}{\partial x^2}+f^{(2)}(t,x)\right), & x\in\omega_{2h_2},\end{cases}$$

$$Jv=(Jv)(t,x)=\sum_{t'=\tau}^{t}\tau\exp\left(\frac{t'-(t+\tau)}{\theta}\right)v(t',x),\quad \widehat{J}v=(Jv)(t+\tau,x),$$

$$\varphi_1(t,x) = \begin{cases} f^{(1)}(t,x), & (t,x) \in \omega_1, \\ (Jf^{(1)})(t,\xi), & t \in \omega_\tau, \end{cases} \qquad \varphi_2(t,x) = f_2(t,x), \quad (t,x) \in \omega_2 \cup \xi,$$

$$y = y(t,x) = \begin{cases} y_1(t,x), & x \in \omega_1, \\ y_2(t,x), & x \in \omega_2, \end{cases}$$

Here we also use non-indexed notation of the difference scheme theory [8].

Note that consistency condition (2.7) is approximated taking into account requirement of the second order of approximation with respect to spatial variable.

For the scheme (2.9) — (2.13) the following statement regarding convergence is fulfilled.

Theorem 2.1 [6; 7]. *Under the conditions* (2.8) *solution of the difference scheme* (2.9) – (2.13) *converges to the solution of the differential problem* (2.1) – (2.7) *with rate* $\mathcal{O}(\tau + h^{3/2})$, *i.e. for the error* $z = y - u$ *the following estimate holds*

$$\|z\| \leqslant c_0(\tau + h^{3/2}), \quad c_0 = \text{const} > 0.$$

Here the grid L_2*-norm* $\|\cdot\|$ *is defined by the following relation*

$$\|z\| = \sum_{x \in \omega_h} z^2 \hbar, \quad \hbar = \begin{cases} h_1 & \text{for} \quad x \in \omega_{1h_1}, \\ 0{,}5(h_1 + h_2) & \text{for} \quad x = \xi, \\ h_2 & \text{for} \quad x \in \omega_{2h_2}. \end{cases}$$

3. DIFFERENCE SCHEMES WITH VARIABLE WEIGHTING FACTORS FOR HYPERBOLIC-PARABOLIC EQUATIONS

Let $\mathbb{R}^2$ be two-dimensional Euclidean space of points $\boldsymbol{x} = (x_1, x_2)$, $\Omega = \{\boldsymbol{x} = (x_1, x_2) : 0 \leqslant x_\alpha \leqslant l_\alpha,\ \alpha = 1,2\}$ be bounded domain in $\mathbb{R}^2$. Suppose that Ω is separated by the straight line $S = \{\boldsymbol{x} = (x_1, x_2) :\ x_1 = \text{const},\ 0 \leqslant x_2 \leqslant l_2\}$ into two non-intersecting subdomains Ω_1 and Ω_2.

Consider the equations

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left(k_1^{(1)}(\boldsymbol{x})\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k_2^{(1)}(\boldsymbol{x})\frac{\partial u}{\partial x_2}\right) + f_1(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in Q_1, \tag{3.1}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x_1}\left(k_1^{(2)}(\boldsymbol{x})\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k_2^{(2)}(\boldsymbol{x})\frac{\partial u}{\partial x_2}\right) + f_2(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in Q_2, \tag{3.2}$$

where $Q_m = \Omega_m \times (0,T)$, $Q = \Omega \times (0,T)$ $(m = 1,2)$. Suppose that the first equation is parabolic in $\overline{Q}_1$ and the second one is hyperbolic in $\overline{Q}_2$, that is, there exist constants $c_1 > 0$ and $c_2 > 0$ such that for any vector $\xi = (\xi_1, \xi_2)$

$$k_1^{(m)}(\boldsymbol{x})\xi_1^2 + k_2^{(m)}(\boldsymbol{x})\xi_2^2 \geq c_m\left(\xi_1^2 + \xi_2^2\right), \quad m = 1,2.$$

For equations (3.1), (1.1) we formulate the following initial-boundary problem: it is required to find the function u such that it is defined on Q and satisfies the equation (3.1) in Q_1, the equation (1.1) in Q_2, as well as the following initial and boundary conditions:

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \quad \frac{\partial u(\boldsymbol{x},0)}{\partial t} = u_1(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega_2, \tag{3.3}$$

$$u\big|_{\partial Q} = 0, \tag{3.4}$$

and consistency conditions on the interface:

$$u|_{\Gamma-0} = u|_{\Gamma+0}, \quad \left(k_1^{(1)} \frac{\partial u}{\partial x_1}\right)\bigg|_{\Gamma-0} = \left(k_1^{(2)} \frac{\partial u}{\partial x_1}\right)\bigg|_{\Gamma+0}. \tag{3.5}$$

Here ∂Q is the side surface of the cylinder Q, $\Gamma = S \times [0,T]$ is the interface. The symbols $\Gamma - 0$, $\Gamma + 0$ denote that the function limit values on Γ are taken from the subdomains Q_1 and Q_2, respectively.

Suppose that the coefficients $k_1^{(m)}(\boldsymbol{x})$ and the right side $f_m(x,t)$ have the first kind discontinuity on the surface Γ and that these functions are smooth outside of the interface. Then there exists a unique weak solution of problem (3.1)—(3.5) (see [3; 4]).

Further let the solution $u(x,t)$ of problem (3.1)—(3.5) be piece-wise smooth. In other words, the solution $u(x,t)$ has all the necessary continuous and bounded derivatives outside the surface Γ and satisfies the consistency conditions (3.5) on Γ.

3.1. Difference scheme with variable weights on non-uniform grid

In $\overline{\Omega}$ there exists a non-uniform grid $\widehat{\overline{\omega}}_h = \{\boldsymbol{x}_{i_1 i_2} = \left(x_1^{i_1}, x_2^{i_2}\right) : \ x_\alpha^{i_\alpha} = x_\alpha^{i_\alpha - 1} + h_\alpha^{i_\alpha}, \ i_\alpha = 1,2,\ldots,N_\alpha - 1, \ x_\alpha^0 = 0, \ x_\alpha^{N_\alpha} = l_\alpha, \ \alpha = 1,2\}$, such as the interface S contains its nodes $\gamma_h = \{\boldsymbol{x}_{p_1 i_2} = \left(x_1^{p_1}, x_2^{i_2}\right) : \ x_1^{p_1} = p_1 h_1^{p_1}, \ \ x_2^{i_2} = x_2^{i_2-1} + h_2^{i_2}, \ i_2 = 1,2,\ldots,N_2 - 1\}$, where $2 \leqslant p_1 \leqslant N_1 - 2$. Denote by $\widehat{\omega}_h$ the set of internal nodes of the grid $\widehat{\overline{\omega}}_h$, and by $\partial\widehat{\omega}_h$ the set of boundary points of this grid. Let $\overline{\omega}_\tau = \{t_j = j\tau, \ j = 0,1,2,\ldots,N_t\}$ be a uniform grid in time, $\overline{\omega} = \widehat{\overline{\omega}}_h \times \overline{\omega}_\tau$ grid in parallelepiped Q. In addition, in the subdomains Q_1, Q_2 we consider we consider the grids $\omega_1 = \widehat{\omega}_{1h} \times \omega_\tau$, $\omega_2 = \widehat{\omega}_{2h} \times \omega_\tau$, where $\widehat{\omega}_{1h} = \{\boldsymbol{x}_{i_1 i_2} = \left(x_1^{i_1}, x_2^{i_2}\right) : \ x_\alpha^{i_\alpha} = x_\alpha^{i_\alpha - 1} + h_\alpha^{i_\alpha}, \ \alpha = 1,2, i_1 = 1,2,\ldots,p_1 - 1, \ i_2 = 1,2,\ldots,N_2 - 1\}$, $\widehat{\omega}_{2h} = \{\boldsymbol{x}_{i_1 i_2} = \left(x_1^{i_1}, x_2^{i_2}\right) : \ x_\alpha^{i_\alpha} = x_\alpha^{i_\alpha - 1} + h_\alpha^{i_\alpha}, \ \alpha = 1,2, \ i_1 = p_1 + 1, p_1 + 2, \ldots, N_1 - 1, \ i_2 = 1,2,\ldots,N_2 - 1\}$.

Note that for approximation of parabolic equation two-level schemes are usually used, and for approximation of hyperbolic equation three-level schemes. Therefore similarly to [10; 11] we approximate the differential problem (3.1) — (3.5) by the following three-level finite-difference scheme

$$y_{\bar{t}t} = \left(a_1 y_{\bar{x}_1}^{(\sigma_1(\boldsymbol{x}),\sigma_2(\boldsymbol{x}))}\right)_{\hat{x}_1} + \left(a_2 y_{\bar{x}_2}^{(\sigma_1(\boldsymbol{x}),\sigma_2(\boldsymbol{x}))}\right)_{\hat{x}_2} + \varphi, \tag{3.6}$$

$$(\boldsymbol{x},t)\in\omega_1,$$

$$y_t=\left(a_1 y_{\bar{x}_1}^{(\sigma_1(\boldsymbol{x}),\sigma_2(\boldsymbol{x}))}\right)_{\hat{x}_1}+\left(a_2 y_{\bar{x}_2}^{(\sigma_1(\boldsymbol{x}),\sigma_2(\boldsymbol{x}))}\right)_{\hat{x}_2}+\varphi, \tag{3.7}$$

$$(\boldsymbol{x},t)\in\omega_1,$$

$$\frac{1}{2\hbar_1}\left(h_1 y_{\bar{t}t}+h_{1+}y_t\right)=\left(a_1 y_{\bar{x}_1}^{(\sigma_1(\boldsymbol{x}),\sigma_2(\boldsymbol{x}))}\right)_{\hat{x}_1} \tag{3.8}$$

$$+\left(a_2 y_{\bar{x}_2}^{(\sigma_1(\boldsymbol{x}),\sigma_2(\boldsymbol{x}))}\right)_{\hat{x}_2}+\varphi,\quad (\boldsymbol{x},t)\in\gamma_h\times\omega_\tau,$$

$$\left.\widehat{y}\right|_{\partial\hat{\omega}_h}=0, \tag{3.9}$$

$$y(\boldsymbol{x},0)=u_0(\boldsymbol{x}),\quad y_t(\boldsymbol{x},0)=\overline{u}_1(\boldsymbol{x}),\quad \boldsymbol{x}\in\widehat{\overline{\omega}}_h, \tag{3.10}$$

with variable weighting factors

$$\sigma_1(\boldsymbol{x})=\begin{cases}\sigma_1^*, & \boldsymbol{x}\in\widehat{\omega}_{1h},\\ 0{,}5(\sigma+\sigma_1^*), & \boldsymbol{x}\in\gamma_h,\\ \sigma, & \boldsymbol{x}\in\widehat{\omega}_{2h}^{+1},\\ 0, & \boldsymbol{x}\in\partial\widehat{\omega}_h^-,\end{cases}\qquad \sigma_2(\boldsymbol{x})=\begin{cases}\sigma_2^*, & \boldsymbol{x}\in\widehat{\omega}_{1h},\\ 0{,}5\sigma_2^*, & \boldsymbol{x}\in\gamma_h,\\ 0, & \boldsymbol{x}\in\widehat{\omega}_{2h}\cup\partial\widehat{\omega}_h.\end{cases}$$

Here $\sigma,\sigma_1^*,\sigma_2^*$ are positive constants, $\partial\widehat{\omega}_h^-=\partial\widehat{\omega}_h\setminus\{x_1=l_1,\ 0<x_2<l_2\}$, $\widehat{\omega}_{2h}^{+1}=\widehat{\omega}_{2h}\cup\{x_1=l_1,\ 0<x_2<l_2\}$. Note that the second initial condition are determined by analogy with the previous sections.

For the scheme with variable weighting factors (3.6) — (3.10) the following statement is valid.

Theorem 3.1. *Let* $k_\alpha^{(m)}(\boldsymbol{x})\in C^3(\Omega_m)$, $\widetilde{u}\in C^4(\Omega_m)$, $f(\boldsymbol{x},t)\in C^3(\Omega_m)$, $\alpha=1,2,\quad m=1,2$. *Then under the conditions*

$$\sigma_1(\boldsymbol{x})\ge\sigma_2(\boldsymbol{x}),\quad \sigma_1(\boldsymbol{x})+\sigma_2(\boldsymbol{x})\ge 0{,}5(1+\varepsilon),$$
$$\boldsymbol{x}\in\widehat{\omega}_h^{+1}=\widehat{\omega}_h\cup\{x_1=l_1,\ 0<x_2<l_2\},$$

solution of the difference scheme (3.6) – (3.10) *converges to the exact solution of differential problem (3.2)* — (3.5) *and the following estimate holds*

$$\max_{t\in\omega_\tau}\|y-u\|_A\leqslant M(\tau+h_{1\max}^2+h_{2\max}^2),\quad h_{\alpha\max}=\max_{1\leqslant i_\alpha\leqslant N_\alpha}h_\alpha^{i_\alpha},\quad \alpha=1,2.$$

Here $M=\mathrm{const}>0$ *and* $Ay=-\sum\limits_{k=1}^{2}(a_k(\boldsymbol{x})y_{\bar{x}_k})_{\hat{x}_k}$ *for* $\boldsymbol{x}\in\widehat{\omega}_h$, $Ay=0$ *for* $\boldsymbol{x}\in\partial\widehat{\omega}_h$.

REFERENCES

[1] J. A. Akilov. *Non-stationary motions of viscoelastic fluids.* Tashkent: FAN, 1982. (in Russian)

[2] V. I. Burenkov. *Sobolev spaces on domains.* Stuttgart, Leipzig: Teubner, 1998.

[3] V. I. Korzyuk. The problem about conjugation of equations of hyperbolic and parabolic types. *Differents. Uravnenia*, **4** (10), 1968, 1855 – 1866.(in Russian)

[4] V. I. Korzyuk. Mixed problem for some nonstationary equations with discontinuous coefficients. *Differents. Uravnenia*, **6** (2), 1970, 343 – 357.(in Russian)

[5] V. I. Korzyuk. The method of energy inequalities and mollifying operators. *Vestnik Belgosuniversiteta. Ser. 1. Fizika, Matematika, Informatika*, **3** 1996, 55 – 71.(in Russian)

[6] V. I. Korzyuk, S. V. Lemeshevsky, and P. P. Matus. Conjugation problem about jointly separate flow of viscoelastic and viscous fluids in a plane horizontal split. *Dokl. NAN Belarusi*, **44** (2), 2000, 5 – 8.

[7] V. I. Korzyuk, S. V. Lemeshevsky, P. P. Matus, and V. N. Shalima. Conjugation problem about jointly separate flow of viscoelastic and viscous fluids in the plane duct. *Mathematical Modelling and Analysis*, **4** 1999, 114 – 123.

[8] A. A. Samarskii. *Theory of difference schemes.* Moscow: Nauka, 1977. (in Russian)

[9] A. A. Samarskii, V. I. Korzyuk, S. V. Lemeshevsky, and P. P. Matus. Finite-difference methods for problem of conjugation of hyperbolic and parabolic equations. *Mathematical Models and Methods in Applied Sciences*, **10** (3), 2000, 361 – 377.

[10] A. A. Samarskii, P. N. Vabishchevich, S. V. Lemeshevsky, and P. P. Matus. Difference schemes for conjugation problem of hyperbolic and parabolic equations. *Sib. Matem. Zhurnal*, **39** (4), 1998, 954 – 962.(in Russian)

[11] A. A. Samarskii, P. N. Vabishchevich, and P. P. Matus. *Difference schemes with operator factors.* Minsk: ZAO COTJ, 1998. (in Russian)

[12] Ya. S. Uflyand. To reference about oscillations spreading in composite electric lines. *Inzh.-Fiz. Zhurnal*, **7** (1), 1961, 89 – 92.(in Russian)

[13] Ya. S. Uflyand and I. B. Chekmariov. Investigation of unsteady flow of conducting fluid in plain channel with moving boundary. *Zhurn. Tekhnicheskoi Fiziki*, **30** (5), 1960, 465 – 471.(in Russian)

SKIRTINGO TIPO LYGČIŲ JUNGIMO PROBLEMA

V.I. Korzyuk, S.V. Lemeshevsky

Darbe nagrinėjama lygčių sistema, kurią sudaro parabolinė ir hiperbolinė lygtys. Pateikiamos kelios šių lygčių jungimo sąlygos bei surastos atitinkamų uždavinių sprendimo vienaties sąlygos. Duotasis uždavinys aproksimuojamas baigtinių skirtumų schema. Įrodyti energetiniai diskrečiojo sprendinio tikslumo įverčiai. Gautieji rezultatai gali būti pritaikyti silpnai suspaudžiamų skysčių tekėjimo modeliavimui, kai kanalo sienelės yra sudarytos iš poringos medžiagos. Kitas taikomasis pavyzdys atsiranda nagrinėjant elektromagnetinių laukų sąveiką su magnetiniais skysčiais.