AN OBJECTIVE MULTI-CRITERIA APPROACH TO OPTIMIZATION USING MOORA METHOD AND INTERVAL GREY NUMBERS

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Abstract. Many real-world problems are complex and/or related to the manifestation of some form of uncertainty and/or prediction. Therefore the use of extended MCDM methods is more appropriate than the use of the other classic decision making methods. These methods are improved by the use of a form of fuzzy or interval grey numbers.

In the field of operational research, during the previous period, numerous MCDM methods were formed, but one newly proposed, the MOORA method, is very specific and yet has no extension. Therefore, in this paper we combine concept of interval grey numbers and MOORA method in order to propose extended MOORA method which will be more appropriate to solve many complex real-world problems.

Keywords: MCDM, MOORA, interval grey numbers, ratio system, reference point, uncertainty.

1. Introduction

Decision-making is the process of finding the best option from all of the feasible alternatives (Jahanshahloo *et al.* 2006, 2009; Liu 2009). In many cases finding, or more precisely determination, of the best option, is based on the impacts of multiple, often conflicting, criteria, which is why this decision-making type is called Multi-Criteria Decision Making (MCDM).

During the second half of the 20th century MCDM was one of the fastest growing areas of operational research and as a result many MCDM methods have been proposed. There are

many ways in which the classification of MCDM methods can be done, but, according to the useful classification proposed by many authors, the MCDM methods are classified into two main categories: Multi-Objective Decision Making (MODM) methods and Multi-Attribute Decision Making (MADM) methods.

Generally speaking, it can be said that MADM selects the best alternative among a finite number of choices, unlike MODM where the best alternative is designed with multiple objectives based on continuous decision variables subject to constraints (Bernroider, Stix 2007), but often the terms MADM, MCDM, and MODM are mixed up or used with the same meaning and therefore the term criteria can be used to denote both objectives and attributes (Ganoulis 2003).

From many of proposed MCDM methods we shall state some of the most prominent, such as: Simple Additive Weighting (SAW) method (MacCrimon 1968), Analytic Hierarchy Process (AHP) method (Saaty 1980), Technique for Ordering Preference by Similarity to Ideal Solution (TOPSIS) method (Hwang, Yoon 1981), Preference Ranking Organisation Method for Enrichment Evaluations (PROMETHEE) method (Brans, Vincke 1985) and ELimination and Choice Expressing REality (ELECTRE) method (Roy 1991). An overview of these methods is presented in Hwang and Yoon (1981), Yoon and Hwang (1995) and Triantaphyllou and Lin (1996).

Besides the above mentioned methods in the literature some recent MCDM methods are often cited. For example, COmplex PRoportional ASsessment (COPRAS) method (Zavadskas *et al*. 1994, 2009a, b; Kaklauskas *et al*. 2010), VIKOR (VIsekriterijumska optimizacija i KOmpromisno Resenje in Serbian, means Multicriteria Optimization and Compromise Solution) method (Opricovic 1998), Additive Ratio Assessment (ARAS) method (Zavadskas, Turskis 2010; Zavadskas *et al*. 2010a; Turskis, Zavadskas 2010), Multi-Objective Optimization on the basis of Ratio Analysis (MOORA) method (Brauers, Zavadskas 2006) and Multi-Objective Optimization by Ratio Analysis plus Full Multiplicative Form (MULTIMOORA) method (Brauers, Zavadskas 2010).

These MCDM methods have been used to solve a variety of problems in different fields which are also published in a numerous of scientific journals. For example, the COPRAS method is successfully applied for defining the utility and market value of a real estate (Kaklauskas *et al*. 2007) and selecting apartment blocks maintenance contractors (Zavadskas *et al*. 2009b), ARAS method is used to evaluate microclimate in office (Zavadskas, Turskis 2010).

Although the MOORA is a newly proposed method, it has been applied to solve many economic, managerial and construction problems. Chakraborty (2010) uses the MOORA method to solve different decision making problems in the real-time manufacturing environment. Kracka *et al*. (2010) applies the MOORA method in construction in order to solve problems related to energy loss in heating buildings. The aim of his research is to create a technique for the selection of external walls and windows of buildings. In the mentioned field Brauers and Zavadskas (Brauers, Zavadskas 2009; Brauers *et al*. 2008b) use the MOORA method for evaluating contractors in the facilities sector. The MOORA method has also been successfully used for determining the best road design alternative (Brauers *et al*. 2008a).

The use of the MOORA method in the economy is proposed by Brauers and Zavadskas (2010, 2008) and Brauers and Ginevicius (2010, 2009). For example, Brauers and Zavadskas

(2010) use the MOORA method for project management in a transaction economy, and Brauers and Ginevicius (2009) use the MOORA method to define the economic policy for balanced regional development in Lithuania.

In the above mentioned MCDM methods the responses of objectives and their significances are presented using the exact (crisp) values, and such methods are called classical, or ordinary, MCDM methods. However, in many real-world decision making problems the responses of objectives and/or their significances cannot be determined precisely. The inability to determine the exact values increases with problems that involve certain forms of predictions. This type of problems is often present in decision-making in economics and management, for example in the cases of financial and economic analysis for new investment.

For such type of problems, which are characterized by uncertainty and imprecision, the use of values which are expressed in the form of intervals instead of exact (crisp) values can be much more appropriate.

Deng (1982) proposed a grey system theory to study uncertain systems, and also introduces the concept of interval grey numbers. Grey system theory provides an efficient approach to solve problems with significant uncertainty, and therefore has been successfully applied in many fields for analysis, modeling and forecasting. For example Wu and Chang (2003) use Grey analysis for forecasting environmental cost allocations; Du and Sheen (2005) use Gray system theory for forming grey prediction model; Karmakar and Mujumdar (2006) develop grey fuzzy optimization model for water quality management of river system; and Haq and Kannan (2007) apply grey system theory for vendor selection in supply chain management.

Many classical MCDM methods are also extended for the use of interval grey numbers because these extensions allow the use of extended MCDM methods to solve far more problems. Some proposed extensions of classical MCDM methods are: Grey TOPSIS (Chen, Tzeng 2004; Lin *et al.* 2008), Grey AHP (Li *et al.* 2010), COPRAS-G (Zavadskas *et al.* 2008) and SAW-G (Zavadskas *et al.* 2010b; Medineckiene *et al.* 2010; Turskis, Zavadskas 2010).

In order to ensure the prerequisites that are necessary for the use of MOORA method with interval grey numbers in this paper we propose an extension of the MOORA method.

The remaining part of the paper is organized as follows: In section 2 some basic elements of the Grey system theory are presented. In section 3 original MOORA method is presented, and in subsection 3.4 the comparison of the MOORA method and some significant MCDM methods is given. In section 4 an extension of the MOORA adopted for the use of interval grey numbers is presented. In section 5 three numerical examples borrowed from literature are considered in order to verify the proposed approach. Finally, section 6 presents conclusions.

2. Basics of Grey system theory

The Grey systems theory is an effective methodology that can be used to solve uncertain problems with partially known information. The basic concept of grey system theory is that all information can be classified into three categories that are labeled with corresponding colors: known information is white, unknown information is black, and the uncertainty information is grey.

The following definitions are the basic related to Grey systems theory, taken from Deng (1985, 1989, 1992), Liu *et al.* (1999) and Liu and Lin (1998, 2006).

A grey number, denoted as $\otimes x$, is such a number whose exact value is unknown, but a range within which the value lies is known. There are several types of grey numbers such as: grey numbers with only upper limits, grey numbers with only lower limits, black and white numbers and so on, but we will focus below on interval grey numbers.

A grey number with known upper $\overline{x}$, and lower $\underline{x}$, bounds but unknown distribution information for x is called interval grey number (Deng 1989):

$$\otimes x = [\underline{x}, \overline{x}] = [x' \in x \mid \underline{x} \le x' \le \overline{x}].$$ (1)

The degree of greyness of an interval number is determined by distance between its bounds. When upper and lower bounds are equal, $\underline{x} = \overline{x}$, interval grey number becomes a white number, i.e. deterministic number. Otherwise, when distance between bounds increases and bounds tend to infinity, $\underline{x} \to -\infty$ and $\overline{x} \to +\infty$, interval grey number becomes a black number.
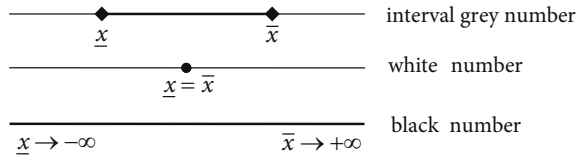

Fig. 1. White, black and interval grey numbers

Basic operations of interval numbers. Let $\otimes x_1 = [\underline{x_1}, \ \overline{x_1}]$ and $\otimes x_2 = [\underline{x_2}, \ \overline{x_2}]$ be two interval grey numbers. The basic operations of grey numbers $\otimes x_1$ and $\otimes x_2$ are defined as follows (Deng 1992; Liu, Lin 2006):

$$\otimes x_1 + \otimes x_2 = [\underline{x_1} + \underline{x_2}, \ \overline{x_1} + \overline{x_2}] \text{ addition;}$$ (2)

$$\otimes x_1 - \otimes x_2 = [\underline{x_1} - \overline{x_2}, \ \overline{x_1} - \underline{x_2}] \text{ subtraction;}$$ (3)

$$\otimes x_1 \times \otimes x_2 = [\underline{x_1}\underline{x_2}, \ \overline{x_1}\overline{x_2}] \text{ multiplication; and}$$ (4)

$$\otimes x_1 \div \otimes x_2 = [\underline{x_1}, \ \overline{x_1}] \times [\frac{1}{\underline{x_2}}, \ \frac{1}{\overline{x_2}}] \text{ division.}$$ (5)

Whitened value. The whitened value of an interval grey number, $\otimes x$, is a deterministic number with its value lying between the upper and lower bounds of interval $\otimes x$. For a given interval grey number $\otimes x = [\underline{x}, \overline{x}]$ the whitened value $x_{(\lambda)}$ can be determined as follows (Liu, Lin 2006):

$$x_{(\lambda)} = \lambda \ \underline{x} + (1 - \lambda) \ \overline{x},$$ (6)

with λ as whitening coefficient and $\lambda \in [0,1]$. Because of its similarity with a popular λ function formula (6) is often shown in the following form:

$$x_{(\lambda)} = (1 - \lambda)\underline{x} + \lambda\overline{x}.$$ (7)

For $\lambda = 0.5$ formula (7) gets the following form:

$$x_{(\lambda=0.5)} = \frac{1}{2}(\underline{x} + \overline{x}).$$ (8)

Signed distance. Let $\otimes x_1 = [\underline{x}_1, \ \overline{x}_1]$ and $\otimes x_2 = [\underline{x}_2, \ \overline{x}_2]$ be two positive interval grey numbers. Then, the distance between $\otimes x_1$ and $\otimes x_2$ can be calculated as signed difference between its centers (Eberly 2007), as is shown below:

$$d(\otimes x_1, \otimes x_2) = \frac{\underline{x}_1 + \overline{x}_1}{2} - \frac{\underline{x}_2 + \overline{x}_2}{2} = \frac{1}{2}[(\underline{x}_1 - \underline{x}_2) + (\overline{x}_1 - \overline{x}_2)].$$ (9)

3. The MOORA method

Multi-Objective Optimization by Ratio Analysis (MOORA) method is introduced by Brauers and Zavadskas (2006) on the basis of previous researches (Brauers 2004a, 2004b).

The method starts with a matrix of responses of different alternatives on different objectives:

$$X = [x_{ij}]_{m \times n},$$ (10)

with: x_{ij} as the response of alternative j on objective or attribute i; $i = 1, 2, ..., n$ as the objectives or the attributes; and $j = 1, 2, ..., m$ as the alternatives.

The MOORA method consists of two parts: the Ratio system and the Reference point approach (Brauers, Zavadskas 2009).

3.1. The Ratio system approach of the MOORA method

Brauers and Zavadskas (2006) prove that the most robust choice for denominator is the square root of the sum of squares of each alternative per objective, and therefore the use of vector normalization method is recommended in order to normalize responses of alternatives. As a result, the following formula proposed by Van Delft and Nijkamp (1977) is used:

$$x_{ij}^* = \frac{x_{ij}}{\sqrt{\sum_{j=1}^{m} x_{ij}^2}},$$ (11)

with: x_{ij} as response of alternative j on objective i; $j = 1, 2, ..., m$; m the number of alternatives; $i = 1, 2, ... n$; n the number of objectives; x_{ij}^* as normalized response of alternative j on objective i; and $x_{ij}^* \in [0, 1]$.

For optimization based on the Ratio system approach of MOORA method, normalized responses are added in case of maximization and subtracted in case of minimization, which can be expressed by the following formula:

$$y_j^* = \sum_{i=1}^{g} x_{ij}^* - \sum_{i=g+1}^{i=n} x_{ij}^*,$$ (12)

with: x_{ij}^* as normalized response of alternative j on objective i; $i = 1, 2, ..., g$ as the objectives to be maximized; $i = g + 1, g + 2, ..., n$ as the objectives to be minimized; $j = 1, 2, ..., m$ as the alternatives; and y_j^* as the overall ranking index of alternative j, $y_j^* \in [-1, 1]$.

After that, the optimal alternative based on the ratio system part A_{RS}^* can be determined using the following formula:

$$A_{RS}^* = \left\{ a_j \middle| = \max_j y_j^* \right\}. \tag{13}$$

3.2. The Reference point approach of the MOORA method

The Reference point approach of the MOORA method is based on the Ratio system and starts from already normalized responses of alternatives, obtained by formula (11).

After considering the most important reference point metrics, Brauers and Zavadskas (2006, 2009), Brauers *et al.* (2008a) and Brauers (2008) emphasize that the min-max metric is the best choice among all of them. Therefore, for optimization based on the reference point approach Brauers and Zavadskas (2006) proposed the following formula:

$$\min_j \left\{ \max_i \left| r_i - x_{ij}^* \right| \right\}, \tag{14}$$

with: r_i as i-th coordinate of the reference point; x_{ij}^* as the normalized response of alternative j on objective i; $i = 1, 2, ..., n$ as the objectives; and $j = 1, 2, ..., m$ as the alternatives.

For further simpler presentations, we will mark distance from an alternative to the reference point with d and therefore formula (14) gets the following form:

$$\min_j \max_i d_{ij}, \tag{15}$$

where:

$$d_{ij} = \left| r_i - x_{ij}^* \right|; \text{ and} \tag{16}$$

$$r_i = \begin{cases} \max_j x_{ij}^* & \text{for objectives to be maximized} \\ \min_j x_{ij}^* & \text{for objectives to be minimized} \end{cases} \tag{17}$$

with: x_{ij}^* as the normalized response of alternative j on objective i; r_i as i-th coordinate of the reference point; d_{ij} as unsigned distance of alternative j to the i-th coordinate of reference point; $i = 1, 2, ..., n$ as the objectives; and $j = 1, 2, ..., m$ as the alternatives.

Based on the Reference point approach of the MOORA method, the optimal alternative A_{RP}^* can be determined using the following formula:

$$A_{RP}^* = \left\{ a_j \middle| = \min_j \max_i d_{ij} \right\}. \tag{18}$$

3.3. The importance given to objectives

When solving a real-world problems using MCDM methods, objectives do not always have the same importance, i.e. some objectives are more important than the others. In order to give more importance to an objective, it could be multiplied with a Significance Coefficient (Brauers, Zavadskas 2009). Importance given to objectives has influence on Ratio system and Reference point approach of the MOORA method.

In the Ratio system approach importance given to objectives is included by modifying the formula (12) which gets the following form:

$$\ddot{y}_j^* = \sum_{i=1}^{g} s_i x_{ij}^* - \sum_{i=g+1}^{i=n} s_i x_{ij}^*, \tag{19}$$

with: s_i as significance coefficient of objective i; $i = 1, 2, ..., g$ as the objectives to be maximized; $i = g+1, g+2, ..., n$ as the objectives to be minimized; $j = 1, 2, ..., m$ as the alternatives; and $\ddot{y}_j^*$ as the overall ranking index of alternative j with respect to all objectives with significance coefficients, $\ddot{y}_j^* \in [-1, 1]$.

After that, the formula (13) still remains to determine the most appropriate alternative based on Ratio system approach of the MOORA method.

As the most effective way to include importance given to objectives into Reference point approach of the MOORA method, we propose to adopt formula (16), which after adoption gets the following form:

$$d_{ij} = s_i \left| r_i - x_{ij}^* \right|, \tag{20}$$

with: s_i as significance coefficient of objective i; x_{ij}^* as the normalized response of alternative j on objective i; r_i as i-th coordinate of the reference point; d_{ij} as distance of alternative j to the i-th coordinate of reference point; $i = 1, 2, ..., n$ as the objectives; and $j = 1, 2, ..., m$ as the alternatives.

After that, the formula (18) still remains without changes for determining the most appropriate alternative based on the Reference point approach of the MOORA method.

3.4. Comparison of the MOORA and some prominent MCDM methods

Researches in the field of MCDM are endless and searching for the best method is still actual. Therefore, in this subsection, after quick review of the most important characteristic of some prominent MCDM methods, we made comparison of the MOORA method and considered methods.

SAW method. The SAW method is probably the simplest, best known and earlier often used MCDM method. Its basic principles are discussed in Churchman and Ackoff (1954) and Triantaphyllou and Lin (1996).

The basic idea of SAW method is that the overall ranking index for each alternative is calculated as the sum of products of its responses and corresponding significance coefficient of objectives, as shown in the following formula:

$$S_j = \sum_{i=1}^{n} x_{ij}^* \, s_i, \tag{21}$$

with: x_{ij}^* as a normalized response of alternative j on objective i; s_i as significance coefficient of objective i; $i = 1,2, \dots n$ as the objectives; $j = 1,2, \dots, m$ as the alternatives; and S_j as overall ranking index of alternative j.

On the basis of S_j, the best alternative is determined using the following formula:

$$A_{SAW}^* = \left\{ a_j \Big| = \max_j S_j \right\}.$$

(22)

The SAW method was used with different normalization procedures. Linear scale transformation - max method is the most frequently used normalization, but there are also other approaches. For example Ginevicius presented the use of SAW method with linear scale transformation - sum method (Ginevicius 2008; Ginevicius, Podvezko 2001, 2006).

In the aggregating function used by the SAW method there is no difference between objectives to be maximized and objectives to be minimized. Therefore, during normalization, the values of objectives to be minimized also must be transformed into the corresponding values of objectives to be maximized. The need for such type of transformations was affected with the formation of numerous normalization procedures and as a result of this still actual question appeared: Which normalization procedure allows to find the best ranking order of alternatives?

TOPSIS method. The TOPSIS method is one of the most widely used multi-criteria method. The basic principle of TOPSIS method is that the best alternative should have the shortest distance from the (positive) ideal solution (PIS) and the farthest distance from the negative-ideal solution (NIS). A relative distance of each alternative from an ideal one is obtained as:

$$C_j = \frac{d_j^-}{d_j^+ + d_j^-},$$

(23)

with: d_j^+ and d_j^- are separation measures of alternative j from the ideal and negative-ideal solution, respectively; C_j as relative distance of alternative j to the ideal solution, $C_j \in [0, 1]$; and $j = 1,2, \dots, m$ as the alternatives. After determining the relative distance for each alternative the best alternative is determined by using the following formula:

$$A_{TPS}^* = \left\{ a_j \Big| = \max_j C_j \right\}.$$

(24)

In the TOPSIS method, the separation measures of each alternative from the PIS and from the NIS are computed using following formulas:

$$d_j^+ = \left(\sum_{i=1}^n (x_{ij}^* s_i - r_i^+)^2 \right)^{0.5}; \text{ and}$$

(25)

$$d_j^- = \left(\sum_{i=1}^n (x_{ij}^* s_i - r_i^-)^2 \right)^{0.5},$$

(26)

where:

$$\left.\begin{array}{l} r_i^+ = \max_j x_{ij}^* \cdot s_i \\ r_i^- = \min_j x_{ij}^* \cdot s_i \end{array}\right\} \text{ for objectives to be maximized; and} \tag{27}$$

$$\left.\begin{array}{l} r_i^+ = \min_j x_{ij}^* \cdot s_i \\ r_i^- = \max_j x_{ij}^* \cdot s_i \end{array}\right\} \text{ for objectives to be minimized,} \tag{28}$$

with: x_{ij}^* as normalized response of alternative j on objective i; s_i as significance coefficient of objective i; r_i^+ and r_i^- as i-th coordinate of PIS and NIS, respectively; d_j^+ and d_j^- as separation measures of alternative j from the PIS and NIS, respectively; $i = 1,2, \dots n$ as the objectives; and $j = 1,2, \dots, m$ as the alternatives.

In the ordinary TOPSIS method vector normalization procedure is used and during performing normalization process there are no need to transform objectives to be minimized into objectives to be maximized. In many extended variants of the TOPSIS method, using fuzzy or interval gray numbers, vector normalization procedure is often replaced by a simpler normalization procedure, usually with linear scale transformation - max method (Wang, Elhag 2006), but there are also some other approaches (Yang, Hung 2007).

In the literature, these are also cases where, in addition to replacement of normalization procedure, transformation of objectives to be minimized into objectives to be maximized is also done. This modification is done in order to provide easier determination of PIS and NIS and does not represent a significant modification which has influence to rank order of alternatives.

VIKOR method. The VIKOR method is based on the idea of ideal and compromise solution and the overall ranking index for each alternative is calculated using the following formula:

$$Q_j = v \frac{(S_j - S^*)}{(S^- - S^*)} + (1 - v) \frac{(R_j - R^*)}{(R^- - R^*)}, \tag{29}$$

where: $S^* = \min_j S_j$; $\quad S^- = \max_j S_j$; $\quad R^* = \min_j R_j$; $\quad R^- = \max_j R_j$, $\tag{30}$

with: Q_j as overall ranking index of alternative j; v as significance of the strategy of the majority of criteria (objectives) whose value is usually set to be 0.5 ; S_j and R_j as the average and the worst (reagent) group score for "the majority of criteria" of alternative j, respectively; S^* and R^* as a maximum of average and worst group scores, respectively; and S^- and R^- as a minimum of average and worst group scores, respectively.

After determining overall ranking index for each alternative, the alternative with smallest overall ranking index has higher priority (rank) and is the most acceptable alternative, as shown in the following formula:

$$A_{VKR}^* = \left\{ a_j \middle| = \min_j Q_j \right\}. \tag{31}$$

Before explaining the meaning of S^- and R^- we consider that development of the VIKOR method started with the following form of L_p metric:

$$L_{p,j} = \left\{ \sum_{i=1}^{n} [s_i(x_i^+ - x_{ij})/(x_i^+ - x_i^-)]^p \right\}^{1/p}, \qquad (32)$$

with[1]: x_{ij} as response of alternative j on objective i; x_i^+ as the most preferable and x_i^- the worst response of all alternatives on objective i; s_i as significance coefficient of objective i; p as metric, $1 \le p \le \infty$; $i = 1,2, \dots n$ as the objectives; and $j = 1,2, \dots, m$ as the alternatives.

The VIKOR method uses two characteristic metrics to formulate ranking measure, $p = 1$ and $p \to \infty$, for which the formula (32) gets the following specific forms:

$$S_j = \sum_{i=1}^{n} s_i(x_i^+ - x_{ij})/(x_i^+ - x_i^-); \text{ for } p = 1, \text{ and} \qquad (33)$$

$$R_j = \max_i \ [s_i(x_i^+ - x_{ij})/(x_i^+ - x_i^-)]; \text{ for } p \to \infty, \qquad (34)$$

where:

$$\left. \begin{array}{l} x_i^+ = \max_j x_{ij} \\ x_i^- = \min_j x_{ij} \end{array} \right\} \text{ for objectives to be maximized; and} \qquad (35)$$

$$\left. \begin{array}{l} x_i^+ = \min_j x_{ij} \\ x_i^- = \max_j x_{ij} \end{array} \right\} \text{ for objectives to be minimized,} \qquad (36)$$

with: x_{ij} as response of alternative j on objective i; x_i^+ as the most preferable and x_i^- the worst response of all alternatives on objective i; s_i as significance coefficient of objective i; p as metric; S_j and R_j as the average and the worst group score of alternative j.

The VIKOR method is similar to the TOPSIS method but there are some significant differences that reflect themselves in the normalization and aggregation phases, i.e. way of determining the overall ranking index. As previously stated, the TOPSIS method uses Vector normalization while the VIKOR method uses linear scale transformation - max min method. However, the main difference appears in the aggregation approaches.

The TOPSIS method introduces the ranking index in formula (23), including the distances from the ideal point and from the negative-ideal point. These distances in TOPSIS are simply summed in formula (23), without considering their relative importance (Opricovic, Tzeng 2004).

The VIKOR method introduces an aggregating function representing the distance from the ideal solution. This ranking index is an aggregation of all criteria (objectives), the relative importance of the criteria (objectives) and a balance between total and individual satisfaction (Opricovic, Tzeng 2004).

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[1]   Some labels used in the formulas proposed by authors of the VIKOR method are changed and harmonized with labels that are used in previously discussed methods and according to Ginevicius (2008).

COPRAS method. In the COPRAS method the priority order of compared alternatives is determined on the basis of their relative weights. The relative weight Q_j of alternative j is calculated by using the following formula:

$$Q_j = S_{+j} + \frac{S_{-\min} \sum_{j=1}^{m} S_{-j}}{S_{-j} \sum_{j=1}^{m} (S_{-\min} / S_{-j})},$$ (37)

where: $S_{+j} = \sum_{i=1}^{g} s_i \cdot x_{ij}^*$; $S_{-j} = \sum_{i=g+1}^{n} s_i \cdot x_{ij}^*$; $S_{-\min} = \min_j S_{-j}$, (38)

with[2]: S_{+j} and S_{-j} as sums of weighted normalized responses of alternative j for objectives to be maximized and objectives to be minimized, respectively; s_i as significance coefficient of objective i; x_{ij}^* as normalized response of alternative j on objective i; $i = 1, 2, \cdots, g$ as the objectives to be maximized; $i = g+1, g+2, \cdots, n$ as the objectives to be minimized; and $j = 1, 2, \ldots, m$ as the alternatives. After determining the relative weight for each alternative, the best alternative is determined by using the following formula:

$$A_{CPS}^* = \left\{ a_j \left| = \max_j Q_j \right. \right\}.$$ (39)

In order to normalize responses of alternatives, COPRAS method uses a linear scale transformation - sum method. Aggregation procedures used in the COPRAS method does not require transformation of responses of alternatives for objectives to be minimized during the normalization process. In order to determine the overall rankings index, weighted normalized ratings of objectives to be maximized are summed while impact of objectives to be minimized, R_j, is included using the second part of formula (37), which is:

$$R_j = \frac{S_{-\min} \sum_{j=1}^{m} S_{-j}}{S_{-j} \sum_{j=1}^{m} (S_{-\min} / S_{-j})}.$$ (40)

The presented procedure of COPRAS method indicates that it can be easily applied for evaluating the alternatives and selecting the most efficient one, with decision maker being completely aware of the physical meaning of the process (Ustinovichius *et al.* 2007).

For more comparison of some of the mentioned MCDM methods see Podvezko (2011), Ginevicius (2008), Ginevicius *et al.* (2008) and Opricovic and Tzeng (2004).

After reviewing most important characteristics of some of the most prominent MCDM methods, we will make some conclusions about the complexity of ranking procedure and approaches applied in the MOORA method.

Complexity. Compared with the previously discussed methods, the MOORA method has slightly complex ranking procedure in comparison with the SAW method but is still less complex in comparison with TOPSIS method. Procedures used to rank the alternatives in the MOORA method are also simpler than procedures used in some other MCDM methods, for example in the VIKOR method.

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[2]   Labels used in formulas are adopted form Kaklauskas *et al.* (2007) and Ginevicius (2008).

Approach. In comparison with other MCDM methods the MOORA method is very specific because it provides two different approaches for ranking alternatives: the Ratio system and the Reference point approach.

Based on the characteristics of the Ratio system approach the MOORA method can be classified as performance-based method, similar as the SAW and COPRAS methods. In comparison with SAW method the MOORA method does not require transformation of objectives to be minimized into objectives to be maximized. As the COPRAS method, the MOORA method differently treats objectives to be maximized and objectives to be minimized in the aggregation stage. Many other MCDM methods also have different treatment of these tupes of objectives, but COPRAS and MOORA methods have relatively simple to use aggregation procedures.

These are not all specific characteristics of the MOORA method. The MOORA method characterizes clear separation between objectives to be maximized and objectives to be minimized, which is particularly evident when the overall ranking index of each alternative is determined by a difference between sums of ratings obtained on the basis of objectives to be maximized and objectives to be minimized. Such approach for determining ranking index of any considered alternative is precise, logically based, relatively easy to use and understandable to decision makers.

Second approach provided by the MOORA method, the Reference point approach, classifies the MOORA method into distance-based methods, similar to the well-known TOPSIS method.

The distance based approach of the MOORA method is based on Minkowski distance metric which can be expressed using the following formula:

$$\min M_j = \left\{ \sum_{i=1}^{n} | r_i - x_{ij}^\star |^p \right\}^{1/p} ; \text{ or} \tag{41}$$

$$\min M_j = \left\{ \sum_{i=1}^{n} s_i^p | r_i - x_{ij}^\star |^p \right\}^{1/p} , \tag{42}$$

when the objectives have different significances, with: r_i as the i-th coordinate of the reference point; $x_{ij}^\star$ as normalized response of alternative j on objective i; s_i as significance coefficient of objective i; p as a parameter that allows different spatial metrics to be used, $p \in [1, \infty]$; and M_j as Minkowski metric for alternative j in n-dimensional space.

The parameter p can take values from the interval $[1, \infty]$, but the most commonly used values are: 1, 2, and ∞. The case when p = 1 is known as the "city-block" or the Manhattan distance. In the case when p = 2, we have a well-known and probably the most commonly used form of distance metrics, the Euclidean distance. The last special case of Minkowski metrics is when $p = \infty$, or more precisely, when p tends to infinity, and then we have the so-called "dominant metric" or max-min metric.

There are many discussions in the literature about which metric gives better results and many discussions about it come from psychological theory. We will not discuss it this time. We will just mention that the TOPSIS method uses $p = 2$ metric, and that there are some attempts to use TOPSIS methods with $p = 1$ metric (Yoon, Hwang 1980; Dasarathy 1976).

We will also mention that the MOORA method uses $p = \infty$ metric which is why the ranking of alternatives based on the Reference point approach is also precise, logically based, relatively easy to use and understandable to decision makers.

With respect to all the facts mentioned above, we conclude that the MOORA method is the most effective and perspective MCDM method and therefore in the remaining part of the paper we proposes an extension of the MOORA method.

4. The Grey MOORA

The procedure of selecting the most appropriate alternative using the MOORA method involves several important stages that should be considered before an extension of the MOORA method with interval grey numbers, and these are:

    stage 1: transforming responses of alternatives into dimensionless values;

    stage 2: determining overall ranking indexes for considered alternatives based on Ratio System part of MOORA method; and

    stage 3: determining distances between considered alternatives and reference point based of the reference point part of MOORA method.

4.1. Stage 1: Transformation into dimensionless values

The first step that should be considered is a way of transforming responses of alternatives into dimensionless values. Compared with other normalization methods, vector normalization method is the most complex. Therefore, in some proposed extensions of other MCDM methods, vector normalization is often replaced by a simpler, usually with a linear transformation - max method (Wang, Elhag 2006). However, this approach does not represent generally adopted rule. For the normalization of responses of alternatives expressed in the form of interval numbers, Jahanshahloo *et al.* (2006) suggested the use of the following formula:

$$\otimes x_{ij}^{*} = \frac{\otimes x_{ij}}{\sqrt{\sum_{j=1}^{m}\left(\underline{x}_{ij}^{2} + \overline{x}_{ij}^{2}\right)}}. \tag{43}$$

Formula (43) provides the appropriate form for normalizing responses of alternatives expressed by interval grey numbers. However, in cases of multi-criteria optimizations which require simultaneously the use of crisp and interval grey numbers, the previously mentioned formula give unsatisfactory results. Therefore, we suggest the use of the following formula:

$$\otimes x_{ij}^{*} = \frac{\otimes x_{ij}}{\sqrt{\frac{1}{2}\sum_{j=1}^{m}\left(\underline{x}_{ij}^{2} + \overline{x}_{ij}^{2}\right)}}. \tag{44}$$

Based on the formula (44), upper and lower bounds of an interval grey number can be determined using the following formulae:

$$\overline{x}^*_{ij} = \frac{\overline{x}_{ij}}{\sqrt{\dfrac{1}{2}\sum_{j=1}^{m}\left(\underline{x}^2_{ij}+\overline{x}^2_{ij}\right)}} \; ; \text{ and} \tag{45}$$

$$\underline{x}^*_{ij} = \frac{\underline{x}_{ij}}{\sqrt{\dfrac{1}{2}\sum_{j=1}^{m}\left(\underline{x}^2_{ij}+\overline{x}^2_{ij}\right)}} \; . \tag{46}$$

Appendix A proves that the formula (44) is a better solution for the normalization of interval grey numbers.

4.2. Stage 2: Determining overall ranking index based on Ratio system approach of the MOORA method

For optimization based on the Ratio system part of the MOORA method we start from the formula:

$$y^*_j = y^+_j - y^-_j , \tag{47}$$

where:

$$y^+_j = \sum_{i\in\Omega^+_C} s_i x^*_{ij} + \sum_{i\in\Omega^+_G} \otimes s_i x^*_{ij} \; ; \tag{48}$$

$$y^-_j = \sum_{i\in\Omega^-_C} s_i x^*_{ij} + \sum_{i\in\Omega^-_G} \otimes s_i x^*_{ij} \; , \tag{49}$$

with: y^*_j as the overall ranking index of alternative j; y^+_j and y^-_j as total sums of maximizing and minimizing responses of alternative j to objectives respectively; s_i as significance coefficient of objective i; x^*_{ij} and $\otimes x^*_{ij}$ as the normalized responses of alternative j on different objectives, which are expressed in the form on crisp or interval grey numbers; Ω^+_C and Ω^+_G as sets of objectives to be maximized expressed in the form on crisp or interval grey numbers; Ω^-_C and Ω^-_G are sets of objectives to be minimized expressed in the form on crisp or interval grey numbers. By replacing formulas (48) and (49) in the formula (47), we get the following formula:

$$y^*_j = \sum_{i\in\Omega^+_C} s_i x^*_{ij} - \sum_{i\in\Omega^-_C} s_i x^*_{ij} + \sum_{i\in\Omega^+_G} \otimes s_i x^*_{ij} - \sum_{i\in\Omega^-_G} \otimes s_i x^*_{ij} \; . \tag{50}$$

Based on the formulas (50), (7) and (9) we get the final and complete formula form:

$$y^*_j = \sum_{i\in\Omega^+_C} s_i x^*_{ij} - \sum_{i\in\Omega^-_C} s_i x^*_{ij} + (1-\lambda)\left(\sum_{i\in\Omega^+_G} s_i \underline{x}^*_{ij} - \sum_{i\in\Omega^-_G} s_i \underline{x}^*_{ij}\right) + \lambda\left(\sum_{i\in\Omega^+_G} s_i \overline{x}^*_{ij} - \sum_{i\in\Omega^-_G} s_i \overline{x}^*_{ij}\right), \tag{51}$$

with: s_i as significance coefficient of objective i; x^*_{ij} as the normalized responses of alternative j on objective i and $i\in\Omega_C$; $\underline{x}^*_{ij}$ and $\overline{x}^*_{ij}$ as the normalized bounds of interval grey number which represents response of alternative j on objective i and $i\in\Omega_G$, respectively; Ω_C and

Ω_G as sets of objectives expressed in the form of crisp or interval grey numbers, respectively; λ as the whitening coefficient; y_j^* as the overall ranking index of alternative j; Ω_C^+ and Ω_G^+ as sets of objectives to be maximized expressed in the form on crisp or interval grey numbers; Ω_C^- and Ω_G^- are sets of objectives to be minimized expressed in the form on crisp or interval grey numbers; $i = 1,2, \ldots n$ as the objectives; and $j = 1,2, \ldots, m$ as the alternatives.

The proposed formula (51) is quite complex but it enables selection of more appropriate alternative, i.e. optimization, in the cases of solving many complex real-world problems such as:

– decision making problems where responses of alternatives can be more appropriately expressed by simultaneous use of crisp and interval grey numbers;
– decision making problems where responses of alternatives can be more appropriately expressed with interval grey numbers, such as problems that require certain estimates and predictions; and
– problems that require investigation of more options in order to choose the most appropriate alternative, i.e. check variants resulting from the optimistic, realistic and pessimistic attitude of decision makers.

In the case of solving complex real-world problems that require simultaneous use of crisp and interval grey numbers, formula (51) provides adequate ability to rank and select the most appropriate alternative.

In the case of solving well-structured problems, the second part of formula (51) which includes the impact of objectives whose responses are expressed using interval grey numbers, has no influence on ranking index and therefore formula (51) can be transformed into following forms:

$$y_j^* = \sum_{i \in \Omega_C^+} x_{ij}^* - \sum_{i \in \Omega_C^-} x_{ij}^* \text{ ; or} \tag{52}$$

$$y_j^* = \sum_{i \in \Omega_C^+} s_i x_{ij}^* - \sum_{i \in \Omega_C^-} s_i x_{ij}^* \text{ ,} \tag{53}$$

when objectives have different significances. The formulae (52) and (53) have the same meanings as formulae (12) and (19), respectively, in original MOORA method.

On the other hand, in the case of solving semi-structured problems, the first part of formula (51) which represents the impact of objectives whose responses are expressed using crisp numbers, has no influence to the overall ranking index and therefore it can be transformed into one of three following forms:

i. when objectives have the same significance:

$$y_j^* = (1 - \lambda) \left(\sum_{i \in \Omega_G^+} \underline{x}_{ij}^* - \sum_{i \in \Omega_G^-} \underline{x}_{ij}^* \right) + \lambda \left(\sum_{i \in \Omega_G^+} \overline{x}_{ij}^* - \sum_{i \in \Omega_G^-} \overline{x}_{ij}^* \right); \tag{54}$$

ii. when the decision maker has no preferences $(\lambda = 0.5)$:

$$y_j^* = \frac{1}{2} \left(\sum_{i \in \Omega_G^+} s_i \underline{x}_{ij}^* - \sum_{i \in \Omega_G^-} s_i \underline{x}_{ij}^* \right) + \frac{1}{2} \left(\sum_{i \in \Omega_G^+} s_i \overline{x}_{ij}^* - \sum_{i \in \Omega_G^-} s_i \overline{x}_{ij}^* \right); \tag{55}$$

iii. and when the decision maker has no preference and objectives have the same significance:

$$y_j^* = \frac{1}{2}\left(\sum_{i\in\Omega_G^+} \underline{x}_{ij}^* - \sum_{i\in\Omega_G^-} \underline{x}_{ij}^*\right) + \frac{1}{2}\left(\sum_{i\in\Omega_G^+} \overline{x}_{ij}^* - \sum_{i\in\Omega_G^-} \overline{x}_{ij}^*\right). \tag{56}$$

During problem solution, i.e. ranking of alternatives, the attitude of the decision makers can lie between pessimistic and optimistic, and the whitening coefficient λ, allows expression of decision makers degree of optimism or pessimism.

In the cases of particularly expressed optimism, the whitening coefficient λ, in accordance with the formula (7), takes higher values $(\lambda \to 1)$ and ranking order of alternatives is mainly based on the upper bounds of intervals with which overall response of each alternative is expressed, $y_{j(\lambda=1)} = \overline{y}_j^*$. On the other hand, in the cases of particularly expressed pessimism, the whitening coefficient λ takes lower values $(\lambda \to 0)$ and ranking order of alternatives is mainly based on lower bounds of the intervals, $y_{j(\lambda=0)} = \underline{y}_j^*$.

4.3. Stage 3: Determining overall ranking index based on Reference point approach of the MOORA method

The most appropriate alternative based on the Reference point approach of the MOORA method when ratings of alternatives are expressed using exact values can be obtained by the formula (15). However, this formula should be adopted in cases when the Reference point approach of the MOORA method is used to solve complex real-world problems. To explain our approach in details, we start from the min-max metric expressed by the formula:

$$\min_j \max_i d_{ij}, \tag{57}$$

with: d_{ij} as distance of alternative *j* to the *i*-th coordinate of reference point.

In the course of solving many complex real-world problems, responses to the objectives are simultaneously expressed using crisp and interval grey numbers. In this case, the reference point cannot be expressed adequately with "simple" point in *n*-dimensional space. We believe that the reference grey point is a more appropriate solution, where coordinates of grey reference point may be crisp or interval grey numbers, depending on type of values which is used to express ratings of alternatives to the corresponding objectives. Therefore, for determining d_{ij} and r_i for objective *i* in different cases, we propose the following:

i. For objective *i* with crisp responses, the correspondent coordinate of the reference grey point is calculated using the formula (17) and distance to the reference point using formula (16) or (20) when objectives have different significances.

ii. For objectives whose responses are expressed using interval grey numbers formulae are more complex, especially when decision makers have opportunity to express their attitudes about optimism or pessimism. For these reasons, we start from the following formulae:

$$d_{ij} = (1-\lambda)\ \underline{d}_{ij} + \lambda\ \overline{d}_{ij} \text{ ; or} \tag{58}$$

$$d_{ij} = s_i\left((1-\lambda)\ \underline{d}_{ij} + \lambda\ \overline{d}_{ij}\right), \tag{59}$$

when objectives have different significances, where:

$$\underline{d}_{ij} = \left| \underline{r}_i - \underline{x}^*_{ij} \right|; \text{ and} \tag{60}$$

$$\overline{d}_{ij} = \left| \overline{r}_i - \overline{x}^*_{ij} \right|, \tag{61}$$

with: λ as whitening coefficient; $\underline{d}_{ij}$ and $\overline{d}_{ij}$ as distances of alternative j to the i-th coordinate of reference grey point; s_i as significance coefficient of objective i; $i = 1, 2, ..., n$ as the objectives; and $j = 1, 2, ..., m$ as the alternatives.

In the proposed approach every coordinate of reference grey point is represented by appropriate interval grey numbers which bounds are determined by using the following formulae:

$$\left. \begin{array}{l} \overline{r}_i = \max\limits_{j} x^*_{ij} \\ \underline{r}_i = \min\limits_{j} x^*_{ij} \end{array} \right\} \text{ for objectives to be maximized; and} \tag{62}$$

$$\left. \begin{array}{l} \overline{r}_i = \min\limits_{j} x^*_{ij} \\ \underline{r}_i = \max\limits_{j} x^*_{ij} \end{array} \right\} \text{ for objectives to be minimized.} \tag{63}$$

Depending on decision makers' preferences, i.e. whitening coefficient value, the formulae (58) and (59) may have the following specific forms:

    i. in the case of extremely pessimistic decision maker attitude, $(\lambda = 0)$:

$$d_{ij(\lambda=0)} = \begin{cases} \underline{d}_{ij} & \text{when objectives have the same significance; or} \\ s_i \, \underline{d}_{ij} & \text{when objectives have different significances.} \end{cases} \tag{64}$$

    ii. in the case of moderate optimism or when the decision maker has no preference, $(\lambda = 0.5)$:

$$d_{ij(\lambda=0.5)} = \begin{cases} (\underline{d}_{ij} + \overline{d}_{ij})/2 & \text{when objectives have the same significance; or} \\ s_i \, (\underline{d}_{ij} + \overline{d}_{ij})/2 & \text{when objectives have different significances.} \end{cases} \tag{65}$$

    iii. and finally in the case of extremely optimistic decision maker attitude, $(\lambda = 1)$:

$$d_{ij(\lambda=1)} = \begin{cases} \overline{d}_{ij} & \text{when objectives have the same significance; or} \\ s_i \, \overline{d}_{ij} & \text{when objectives have different significances.} \end{cases} \tag{66}$$

Appendix B proves that the formula (59) is a better solution for determining distances of alternatives to the reference point in Reference point approach of the MOORA when interval grey numbers are used.

5. Numerical examples

In this section we consider three numerical examples in order to perform a detailed and precise demonstration of the proposed approach. These examples are taken from literature for the purpose of comparison results obtained by using proposed approach and early published results.

The first example shows the use of proposed approach with interval grey numbers. The second example is used to demonstrate application of proposed approach with simultaneous use of crisp and interval grey number. Finally, the third example is used to demonstrate the use of whitening coefficient λ, and highlights the advantages which may be obtained by using whitening coefficient.

5.1. Example I – The use of the MOORA method with interval grey numbers

To demonstrate the use of the MOORA method with interval grey numbers, we choose one well-known example of selection of the effective dwelling house walls using COPRAS-G method, presented in Zavadskas *et al.* (2008). The selected objectives, significance coefficients, optimization directions and responses of considered alternatives are shown in Table 1.

Table 1. Initial decision-making matrix with values expressed using intervals grey numbers

	Objectives								
	C_1		C_2		C_3		C_4		C_5
	Durability of walls		Thermal transmittance		The estimated cost of m² walls		Weight of m² walls		Human work expenditures
	cycles		W/m K		LTL		kg		hour/m²
Significance	0.21		0.33		0.26		0.09		0.11
Optimization	max		min		min		min		min
Alternatives	$\underline{x}_{1j}$	$\overline{x}_{1j}$	$\underline{x}_{2j}$	$\overline{x}_{2j}$	$\underline{x}_{3j}$	$\overline{x}_{3j}$	$\underline{x}_{4j}$	$\overline{x}_{4j}$	$\underline{x}_{5j}$	$\overline{x}_{5j}$
A_1	75	100	0.22	0.25	72.08	94.71	590	652	4.60	4.60
A_2	75	100	0.22	0.25	89.01	100.93	596	625	4.60	4.60
A_3	75	100	0.21	0.25	80.32	96.42	581	604	4.60	4.60
A_4	25	25	0.24	0.27	67.76	98.1	455	479	4.55	5.01

Objectives used in this example have different units of measures and therefore responses of alternatives must be transformed into non-dimensional values before aggregation procedure is used. This transformation, i.e. normalization, is done by using formula (44), or more precisely: values which represent the upper bounds of interval grey numbers are normalized using formula (45) while values which represents the lower bounds of interval grey numbers are normalized using formula (46). Then normalized decision-making matrix is presented in Table 2.

Table 2. Normalized decision-making matrix

	Objectives									
	C_1		C_2		C_3		C_4		C_5	
Significance	0.21		0.33		0.26		0.09		0.11	
Optimization	max		min		min		min		min	
Alternatives	$\underline{x}_{1j}^{*}$	$\overline{x}_{1j}^{*}$	$\underline{x}_{2j}^{*}$	$\overline{x}_{2j}^{*}$	$\underline{x}_{3j}^{*}$	$\overline{x}_{3j}^{*}$	$\underline{x}_{4j}^{*}$	$\overline{x}_{4j}^{*}$	$\underline{x}_{5j}^{*}$	$\overline{x}_{5j}^{*}$
A_1	0.488	0.651	0.460	0.523	0.411	0.540	0.512	0.566	0.495	0.495
A_2	0.488	0.651	0.460	0.523	0.508	0.576	0.517	0.542	0.495	0.495
A_3	0.488	0.651	0.439	0.523	0.458	0.550	0.504	0.524	0.495	0.495
A_4	0.163	0.163	0.502	0.565	0.387	0.560	0.395	0.416	0.490	0.539

On the basic of data shown in Table 2, the use of formula (55) generates the results shown in Table 3.

Table 3. The ranking results obtained using extended Ratio system part of the MOORA method

Alternatives	$i \in \Omega_G^+$		$i \in \Omega_G^-$				Results		
	$\sum s_i \underline{x}_{ij}$	$\sum s_i \overline{x}_{ij}$	$\sum s_i \underline{x}_{ij}$	$\sum s_i \overline{x}_{ij}$	$\sum_{i \in \Omega_G^+} s_i \underline{x}_{ij} - \sum_{i \in \Omega_G^-} s_i \underline{x}_{ij}$	$\sum_{i \in \Omega_G^+} s_i \overline{x}_{ij} - \sum_{i \in \Omega_G^-} s_i \overline{x}_{ij}$	$y_{j(\lambda=0.5)}^{*}$	Rank
---	---	---	---	---	---	---	---	---
A_1	0.103	0.137	0.359	0.418	−0.257	−0.282	−0.269	1
A_2	0.103	0.137	0.385	0.426	−0.282	−0.289	−0.286	3
A_3	0.103	0.137	0.364	0.417	−0.261	−0.281	−0.271	2
A_4	0.034	0.034	0.356	0.429	−0.321	−0.394	−0.358	4

In order to verify obtained ranking results, the comparative review of ranking results obtained using COPRAS-G method (adopted from Zavadskas *et al.* 2008) and ranking results obtained by proposed approach is shown in Table 4.

Table 4. Ranking results obtained using COPRAS-G method and proposed approach

Alternatives	COPRAS-G		MOORA	
	Q_j	Rank	y_j^{*}	Rank
A_1	0.528	1	−0.269	1
A_2	0.512	3	−0.286	3
A_3	0.526	2	−0.271	2
A_4	0.434	4	−0.358	4

As can be seen, the rank order obtained using COPRAS-G method and extended Ratio system approach of the MOORA method are the same.

Ranking results obtained using formula (51) for characteristic values of whitening coefficient λ, are shown in Table 5.

Table 5. Ranking results obtained using extended Ratio system part of the MOORA method and different values of λ

λ	0		0.5		1	
	y_j^*	Rank	y_j^*	Rank	y_j^*	Rank
A_1	−0.257	1	−0.269	1	−0.282	2
A_2	−0.282	3	−0.286	3	−0.289	3
A_3	−0.261	2	−0.271	2	−0.281	1
A_4	−0.321	4	−0.358	4	−0.394	4

The remaining part of this consideration shows the use of extended Reference point approach of the MOORA method. Extended Reference point approach of the MOORA method starts from normalized responses of alternatives on objectives.

Using data shown in Table 2 and formulae (62) and (63), we determine coordinates of reference grey point, which are shown in Table 6 where are also shown distances of any alternative to reference grey point obtained using formulae (60) and (61).

Table 6. Reference grey point and distances to reference grey point

	Objectives									
	C_1		C_2		C_3		C_4		C_5	
Significance	0.21		0.33		0.26		0.09		0.11	
Optimization	max		min		min		min		min	
$\otimes r$	$\underline{r_1}$	$\overline{r_1}$	$\underline{r_2}$	$\overline{r_2}$	$\underline{r_3}$	$\overline{r_3}$	$\underline{r_4}$	$\overline{r_4}$	$\underline{r_5}$	$\overline{r_5}$
Reference point	0.488	0.651	0.439	0.523	0.387	0.540	0.395	0.416	0.490	0.495
Alternatives	$\underline{d_{1j}}$	$\overline{d_{1j}}$	$\underline{d_{2j}}$	$\overline{d_{2j}}$	$\underline{d_{3j}}$	$\overline{d_{3j}}$	$\underline{d_{4j}}$	$\overline{d_{4j}}$	$\underline{d_{5j}}$	$\overline{d_{5j}}$
A_1	0.000	0.000	0.021	0.000	0.025	0.000	0.117	0.150	0.005	0.000
A_2	0.000	0.000	0.021	0.000	0.121	0.035	0.122	0.127	0.005	0.000
A_3	0.000	0.000	0.000	0.000	0.072	0.010	0.109	0.108	0.005	0.000
A_4	0.325	0.488	0.063	0.042	0.000	0.019	0.000	0.000	0.000	0.044

After the use of formula (59), we can determine resulting distance of alternative j from the i-th coordinate of reference point. For example, the resulting distances of any alternative to the reference point for $\lambda = 0.5$ have values shown in Table 7. The resulting distance with highest value for any alternative and rank order is also shown in Table 7.

Table 7. Distances of any alternative to reference point, for $\lambda = 0.5$

	Objectives						
	C_1	C_2	C_3	C_4	C_5		
Alternatives	d_{1j}	d_{2j}	d_{3j}	d_{4j}	d_{5j}	$\max_i d_{ij}$	Rank
A_1	0.000	0.003	0.003	0.012	0.000	0.012	2
A_2	0.000	0.003	0.020	0.011	0.000	0.020	3
A_3	0.000	0.000	0.011	0.010	0.000	0.011	1
A_4	0.085	0.017	0.003	0.000	0.002	0.085	4

Ranking results obtained using formula (59) for characteristic values of whitening coefficient λ, are shown in Table 8.

Table 8. Ranking results obtained using extended Ratio system approach of the MOORA method and different values of λ

λ	0		0.5		1	
Alternatives	$\max\limits_{i} d_{ij}$	Rank	$\max\limits_{i} d_{ij}$	Rank	$\max\limits_{i} d_{ij}$	Rank
A_1	0.011	1	0.012	2	0.116	1
A_2	0.032	3	0.020	3	0.116	1
A_3	0.019	2	0.011	1	0.116	1
A_4	0.068	4	0.085	4	0.129	4

5.2. Example II – The use of the MOORA method with simultaneous use of crisp and interval grey numbers

The second example which will be considered is borrowed from Medineckiene *et al.* (2010). This example is chosen because it combines the use of crisp and interval grey numbers for representing responses of considered alternatives on objectives. This example is also characteristic because it use SAW-G method, i.e. extended SAW method adopted for use of grey interval numbers.

The initial data that is required for ranking alternatives is shown in Table 9.

Table 9. Initial decision-making matrix with values expressed using crisp and intervals grey numbers

	Objectives										
	C_1	$\otimes C_2$		$\otimes C_3$		C_4	C_5	C_6	C_7	$\otimes C_8$		C_9
Significance	0.078	0.038		0.3900		0.143	0.022	0.015	0.008	0.252		0.053
Optimization	min	min		max		min	min	min	min	min		min
Alternatives	x_{1j}	$\underline{x}_{2j}$	$\overline{x}_{2j}$	$\underline{x}_{3j}$	$\overline{x}_{3j}$	x_{4j}	x_{5j}	x_{6j}	x_{7j}	$\underline{x}_{8j}$	$\overline{x}_{8j}$	x_{9j}
A_1	46400	6	8	70	80	84400	188	16.04	4.86	58000	78300	6810
A_2	43500	6	8	70	80	56600	126	10.76	3.26	63800	75400	5680
A_3	40600	3	6	60	80	69600	155	13.23	3.99	58000	74500	9340

Normalized responses, shown in Table 10, are obtained by using formula (11) when ratings are expressed using exact values and formulae (45) and (46) when ratings are expressed using interval grey numbers.

Table 10. Normalized decision-making matrix

				Objectives							
	C_1	$\otimes C_2$		$\otimes C_3$		C_4	C_5	C_6	C_7	$\otimes C_8$	C_9
Significance	0.078	0.038		0.3900		0.143	0.022	0.015	0.008	0.252		0.053
Optimization	min	min		max		min	min	min	min	min		min
Alternatives	x_{1j}^*	$\underline{x}_{2j}^*$	$\overline{x}_{2j}^*$	$\underline{x}_{3j}^*$	$\overline{x}_{3j}^*$	x_{4j}^*	x_{5j}^*	x_{6j}^*	x_{7j}^*	$\underline{x}_{8j}^*$	$\overline{x}_{8j}^*$	x_{9j}^*
A_1	0.615	0.542	0.723	0.548	0.627	0.685	0.685	0.685	0.686	0.489	0.660	0.529
A_2	0.576	0.542	0.723	0.548	0.627	0.460	0.459	0.460	0.460	0.538	0.635	0.441
A_3	0.538	0.271	0.542	0.470	0.627	0.565	0.565	0.565	0.563	0.489	0.628	0.725

The ranking results obtained using extended Ratio system approach of the MOORA methods are shown in Table 11. Rank order shown in Table 11 is equal to rank order in Medineckiene *et al.* (2010), and it is $A_2 \succ A_3 \succ A_1$.

Table 11. The ranking results obtained using extended Ratio system approach of the MOORA method

	Crisp			Grey						
	I	II	III	IV	V	VI	VII	VIII	IX	X
Alternatives	$i \in \Omega_C^+$	$i \in \Omega_C^-$	I-II	$i \in \Omega_G^+$		$i \in \Omega_G^-$		$\lambda = 0.5$	III+VIII	
	$\sum s_i x_{ij}$	$\sum s_i x_{ij}$	y_{Cj}^*	$\sum s_i \underline{x}_{ij}$	$\sum s_i \overline{x}_{ij}$	$\sum s_i \underline{x}_{ij}$	$\sum s_i \overline{x}_{ij}$	y_{Gj}^*	$y_{Cj}^* + y_{Gj}^*$	Rank
A_1	0.000	0.205	−0.205	0.214	0.244	0.144	0.194	0.060	−0.144	3
A_2	0.000	0.155	−0.155	0.214	0.244	0.156	0.188	0.057	−0.097	1
A_3	0.000	0.187	−0.187	0.183	0.244	0.133	0.179	0.058	−0.129	2

For determining rank order based on extended Reference point approach of the MOORA method, using data shown in Table 10 and formulae (62) and (63), we determine coordinates of reference grey point, which are shown in Table 12. Table 12 also shows distances of any alternative to reference grey point obtained using formulas (60) and (61).

Table 12. Reference grey point and distances to reference grey point

					Objectives							
	C_1	$\otimes C_2$		$\otimes C_3$		C_4	C_5	C_6	C_7	$\otimes C_8$		C_9
Significance	0.078	0.038		0.39		0.143	0.022	0.015	0.008	0.252		0.053
Optimization	min	min		max		min	min	min	min	min		min
$\otimes r$	r_{1j}	$\underline{r}_{2j}$	$\overline{r}_{2j}$	$\underline{r}_{3j}$	$\overline{r}_{3j}$	r_{4j}	r_{5j}	r_{6j}	r_{7j}	$\underline{r}_{8j}$	$\overline{r}_{8j}$	r_{9j}
Reference point	0.538	0.271	0.542	0.548	0.627	0.460	0.459	0.460	0.460	0.538	0.271	0.542
Alternatives	d_{1j}	$\underline{d}_{2j}$	$\overline{d}_{2j}$	$\underline{d}_{3j}$	$\overline{d}_{3j}$	d_{4j}	d_{5j}	d_{6j}	d_{7j}	$\underline{d}_{8j}$	$\overline{d}_{8j}$	d_{9j}
A_1	0.077	0.271	0.181	0.000	0.000	0.226	0.226	0.226	0.226	0.000	0.032	0.088
A_2	0.038	0.271	0.181	0.000	0.000	0.000	0.000	0.000	0.000	0.049	0.008	0.000
A_3	0.000	0.000	0.000	0.078	0.000	0.106	0.106	0.106	0.103	0.000	0.000	0.284

Resulting distances of any alternative to i-th coordinate of the reference grey point obtained by formula (59) and $\lambda = 0.5$ are shown in Table 13. Table 13 also shows the resulting distance with highest value for any alternative and rank order obtained on them.

Table 13. Distances of any alternative to reference point, for $\lambda = 0.5$

Alternatives	C_1 d_{1j}	$\otimes C_2$ d_{2j}	$\otimes C_3$ d_{3j}	C_4 d_{4j}	C_5 d_{5j}	C_6 d_{6j}	C_7 d_{7j}	$\otimes C_8$ d_{8j}	C_9 d_{9j}	$\max\limits_i d_{ij}$	Rank
A_1	0.006	0.009	0.000	0.032	0.005	0.003	0.002	0.004	0.005	0.032	3
A_2	0.003	0.009	0.000	0.000	0.000	0.000	0.000	0.007	0.000	0.009	1
A_3	0.000	0.000	0.015	0.015	0.002	0.002	0.001	0.000	0.015	0.015	2

As can bee seen from Table 12 rank order of alternatives again is equal to rank order in Medineckiene *et al.* (2010), and hi is $A_2 \succ A_3 \succ A_1$.

5.3. Example III – The use of the MOORA method with interval grey numbers and whitening coefficient

Finally, our proof of the proposed approach ends with using the third example which is also adopted from literature. The example borrowed from Zavadskas *et al.* (2009a) is chosen because it clearly shows a very important feature which interval grey numbers provides. By giving more importance to lower or upper bounds of interval grey numbers, decision makers may express their preferences. In the case of solving many real-world problems which is usually characterized with some form of uncertainty this feature may be very important. In some cases decision maker can analyze the problem and choose the most appropriate alternative simple by varying values of whitening coefficient.

Table 14 shows the data required for demonstration of the mentioned characteristics of interval grey numbers.

Table 14. Initial decision-making matrix

	Objectives							
	C_1		C_2		C_3		C_4	
Significance	0.15		0.4		0.2		0.25	
Optimization	max		max		max		min	
Alternatives	$\underline{x}_{1j}$	$\overline{x}_{1j}$	$\underline{x}_{2j}$	$\overline{x}_{2j}$	$\underline{x}_{3j}$	$\overline{x}_{3j}$	$\underline{x}_{4j}$	$\overline{x}_{4j}$
A_1	64	85	50	55	60	80	75	80
A_2	57	81	52	56	62	76	70	75
A_3	61	78	55	58	53	61	70	80
A_4	59	93	54	62	55	72	80	90
A_5	63	89	61	68	54	63	65	75

In this example we start from weighted normalized decision making matrix, which is shown in Table 15 where initial data from Table 14 are normalized using formulae (45) and (46) and then multiplied with appropriate significance coefficients.

Table 15. Weighted-normalized decision-making matrix with values expressed using intervals grey numbers

	Objectives							
	C_1		C_2		C_3		C_4	
Optimization	max		max		max		min	
Alternatives	$s_1 \underline{x}_{1j}^*$	$s_1 \overline{x}_{1j}^*$	$s_2 \underline{x}_{2j}^*$	$s_2 \overline{x}_{2j}^*$	$s_3 \underline{x}_{3j}^*$	$s_3 \overline{x}_{3j}^*$	$s_4 \underline{x}_{4j}^*$	$s_4 \overline{x}_{4j}^*$
A_1	0.058	0.077	0.156	0.172	0.084	0.111	0.110	0.117
A_2	0.052	0.073	0.162	0.175	0.086	0.106	0.103	0.110
A_3	0.055	0.071	0.172	0.181	0.074	0.085	0.103	0.117
A_4	0.053	0.084	0.169	0.193	0.077	0.100	0.117	0.132
A_5	0.057	0.081	0.190	0.212	0.075	0.088	0.095	0.110

Table 16 shows ranking results based on extended Ratio system approach of the MOORA method, obtained on the basis of data from Table 15, formula (55) and characteristic values of whitening coefficient.

Table 16. The ranking results obtained using extended Ratio system approach of the MOORA method

	$i \in \Omega_G^+$		$i \in \Omega_G^-$		$\lambda = 0$		$\lambda = 0.5$		$\lambda = 1$	
Alternatives	$\sum s_i \underline{x}_{ij}$	$\sum s_i \overline{x}_{ij}$	$\sum s_i \underline{x}_{ij}$	$\sum s_i \overline{x}_{ij}$	y_j^*	Rank	y_j^*	Rank	y_j^*	Rank
A_1	0.298	0.360	0.110	0.117	0.188	4	0.216	3	0.243	4
A_2	0.300	0.354	0.103	0.110	0.197	3	0.221	2	0.244	3
A_3	0.301	0.337	0.103	0.117	0.198	2	0.209	5	0.220	5
A_4	0.299	0.377	0.117	0.132	0.182	5	0.214	4	0.245	2
A_5	0.322	0.381	0.095	0.110	0.227	1	0.249	1	0.271	1

In order to verify proposed approach, Table 17 shows comparative review of ranking results obtained by using COPRAS-G method (adopted from Zavadskas *et al.* 2009a) and by using extended Ratio system approach of the MOORA method.

As can be seen, the best and the worst ranked alternatives obtained using COPRAS-G method and extended Ratio system approach of the MOORA method are the same in all cases of ranking (pessimistic, moderate and optimistic).

Rank order of other ranked alternatives is not equal, but it is quite understandable because values of ranking indexes are obtained by the use of aggregation procedures which have significant differences.

Table 17. Ranking results obtained using COPRS-G method and proposed approach

	Rank					
	Pessimistic		Moderate		Optimistic	
Alternatives	COPRAS-G	MOORA	COPRAS-G	MOORA	COPRAS-G	MOORA
A_1	4	4	3 = 4	4	4	3
A_2	2	3	2	3	3	2
A_3	3	2	5	5	5	5
A_4	5	5	3 = 4	2	2	4
A_5	1	1	1	1	1	1

6. Conclusion

Many MCDM methods have been proposed and successfully applied for the solution of numerous problems during past decades. However, many real-world decision making problems are related with uncertainties and/or some form of predictions and therefore their solutions can be much better determined using an extended MCDM method which is adapted for the use of fuzzy or interval grey numbers.

The MOORA method is newly proposed MCDM method but it provides two logically based, relatively easy to use and understandable to decision maker approaches for selecting the most appropriate alternative. It still has no extension and therefore, in this paper, we propose an extension of the MOORA method. In our extension both approaches of the ordinary MOORA method, the Ratio system and the Reference point approach, are adapted for the use of interval grey numbers. Proposed extension of the MOORA method enables relatively simple and effective solution of complex real world problems, which presented procedure, results obtained from considered numerical examples and comparison with known results adopted from literature also proves.

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Appendix A: Is the square root of the one half of the sums of squares of the lower and upper limits of the grey interval numbers that represent performances of each alternative per objectives the best choice for the denominator in the MOORA method?

In order to prove the proposed approach we will start from the very simple example. Suppose that it is necessary to rank two alternatives, denoted by A_1 and A_2, based on two criteria, C_1 and C_2, which have the equal significance. Suppose also that both objectives are benefit type and therefore the optimization direction is maximization.

The initial data required for ranking mentioned alternatives are shown in Table 18. Data obtained after normalizing responses using formula (11) and ranking alternatives based on the Ratio system and Reference point approaches of the MOORA method are shown in Table 19.

Table 18. The initial data

	C_1	C_2
	max	max
A_1	3	5
A_2	5	3

Table 19. The rank order of alternatives on the basis of Ratio System and Reference Point approaches

	Data				Results			
	Initial		Normalized		Ratio System		Reference Point	
	C_1	C_2	C_1	C_2				
	max	max	max	max	$x_{1j}^* + x_{2j}^*$	Rank	$\max(x_{1j}^*, x_{2j}^*)$	Rank
	x_{1j}	x_{2j}	x_{1j}^*	x_{2j}^*				
A_1	3	5	0.51	0.86	1.37	1	0.86	1
A_2	5	3	0.86	0.51	1.37	1	0.86	1
Norm	5.83	5.83						

As can be seen from the Table 19, the results, i.e. rank order of alternatives obtained on the basis of Ratio System and Reference Point parts of the MOORA method is equal.

To provide evidence that the formula (44) is more appropriate for normalization we will make a slight modification of the initial data and responses of alternatives on objective C_2 are presented with interval grey numbers, where the upper and lower bounds of the interval are equal. These data are shown in Table 20.

Table 20. The modified initial data

	C_1	C_2	
	x_{1j}	$\underline{x}_{2j}$	$\overline{x}_{2j}$
	max	max	max
A_1	3	5	5
A_2	5	3	3

Table 21 contains two calculations that vary according to the formula that was used in order to normalize initial data presented with interval gray numbers.

In the **Case I** responses of alternatives expressed using crisp numbers are normalized using the formula (11) while responses expressed with interval grey numbers is normalized using the formula (43). The overall ranking index of any of considered alternative, based on Ratio System part of the MOORA method, is calculated according to the formula (51) and $\lambda = 0.5$.

In the **Case II** instead of the formula (43) the formula (44) is used.

Table 21. The rank order of alternatives on the basis of Ratio System and Reference Point approaches[3]

	Data						Results					
	Initial			Normalized			Ratio System		Reference Point				
	C_1	C_2		C_1	C_2		(51)	Rank	C_1	C_2		$\max\limits_i d_{ij}$	Rank
	x_{1j}	$\underline{x}_{2j}$	$\overline{x}_{2j}$	x^*_{1j}	$\underline{x}^*_{2j}$	$\overline{x}^*_{2j}$		max	d_{1j}	$\underline{d}_{2j}$	$\overline{d}_{2j}$		min
Case I													
A1	3	5	5	0.51	0.61	0.61	1.12	2	0.34	0.25	0.25	0.34	1
A2	5	3	3	0.86	0.36	0.36	1.22	1	0.00	0.49	0.49	0.49	2
Norm – RP[3]	5.83	8.25	8.25	0.86	0.36	0.36							
Case II													
A1	3	5	5	0.51	0.86	0.86	1.37	1	0.34	0.00	0.00	0.34	1
A2	5	3	3	0.86	0.51	0.51	1.37	1	0.00	0.34	0.34	0.34	1
Norm – RP	5.83	5.83	5.83	0.86	0.86	0.86							

As can be seen from the Table 21 Formula (44) provides more appropriate way of normalization when the decision problem requires the simultaneous use of crisp and interval grey numbers.

Appendix B: How to measure the distance between an interval grey number and the reference point?

The procedure for determining the distance between an interval grey number and reference point is more complex than the procedure used for determining distance between an crisp number and reference point and it also initiates three dilemmas. The first dilemma is

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[3] Table rows that contains the norms in columns with initial data and the reference points in columns with normalized data.

which characteristic of an interval grey number in the Reference Point part of the MOORA method best represents the interval grey number: its upper or lower bound, mean, or other characteristics.

The second dilemma is how to determine the reference point when responses of alternatives on objectives are expressed using interval grey numbers or when responses are expressed with simultaneous use of crisp and interval grey numbers? This dilemma also initiates the third, very important, dilemma: What is really reference point in cases of using interval grey numbers? Just crisp point in an n-dimensional space or gray reference point, i.e. grey body - ellipsoid, in n-dimensional space?

The use of only one bound of an interval grey number do not allows adequate representation of interval in all possible cases of optimization. In order to prove this we will start with following simple example.

In this example we do not start from responses of alternatives on objectives because those have impact to the denominator used in formula applied for its normalization, that is why such example may be complex and impractical for further explanation. Therefore we start from the normalized decision-making matrix shown in the Table 22.

Table 22. The initial data and rank results rank obtained by using Reference Point Part of the MOORA method

	Data				Results				
	Normalized				Reference Point				
	C1 max	C2 max			C_1	C_2		$\max_i d_{ij}$	Rank
Case I									
	x^*_{1j}	x^*_{2j}		x^*_m	d_{1j}	d_{2j}			
A_1	0.3	0.5		0.5	0.2	0.0		0.2	1
A_2	0.5	0.3		0.3	0.0	0.2		0.2	1
$\max_j$	0.5	0.5							
Case II									
	x^*_{1j}	$\underline{x}^*_{2j}$	$\overline{x}^*_{2j}$	x^*_m	d_{1j}		$\overline{d}_{2j}$		
A_1	0.3	0.2	0.8	0.5	0.2	0.0	0.0	0.2	2
A_2	0.5	0.2	0.4	0.3	0.0	0.0	0.4	0.4	1
$\max_j$	0.5		0.8	0.5					
Case III									
	x^*_{1j}	$\underline{x}^*_{2j}$	$\overline{x}^*_{2j}$	x^*_m	d_{1j}	$\underline{d}_{2j}$	$\overline{d}_{2j}$	$d_{2j(\lambda=0.5)}$	
A_1	0.3	0.2	0.8	0.5	0.2	0.0	0.0	0.0	0.2	1
A_2	0.5	0.2	0.4	0.3	0.0	0.0	0.4	0.2	0.2	1
$\max_j$	0.5	0.2	0.8	0.5					

Shown example in the Table 22 is composed of three basic parts in which two alternatives, A_1 and A_2, are ranked under different conditions based on two criteria C_1 and C_2, with equal significance and the same direction of optimization.

In the Case I responses of alternatives to the objectives C_1 and C_2 are represented by crisp numbers. In the Case II and Case III responses of alternatives to the objective C_2 are represented by interval grey numbers, where interval bounds are formed in a manner that ensures that interval mean is the same as corresponding exact value in Case I of our example.

Ranking results of the alternatives obtained in Case I and Case II are not equal. It is expected because changes of interval bounds can (may) affect the reference point coordinates.

Probably the easiest way to rank alternatives is to represent interval grey numbers with its mean values and to perform the remaining part of the optimization procedure as in the case of applying crisp numbers. But in this approach advantages of using interval grey numbers significantly decreases. Interval grey numbers may be used for more adequate representation of decision making problems but their use will not be reflected on optimization based on Reference point approach of the MOORA method.

Application of interval numbers is also investigated with other methods, especially with TOPSIS method. One of the characteristic approaches to the application of interval grey numbers and TOPSIS methods may be approach used in Jahanshahloo *et al.* (2006).

Unfortunately, results of research obtained with interval grey numbers and other MCDM methods can not be directly applied to the MOORA method. For example, MOORA method uses the min-max metric and distances to the reference point while TOPSIS method uses Euclidean distances to the ideal and anti-ideal points.

To ensure effective use of interval grey numbers and MOORA method for determining distances of each alternative to the reference point, we will start from the formula (58) or formula (59) when the objectives have different significances.
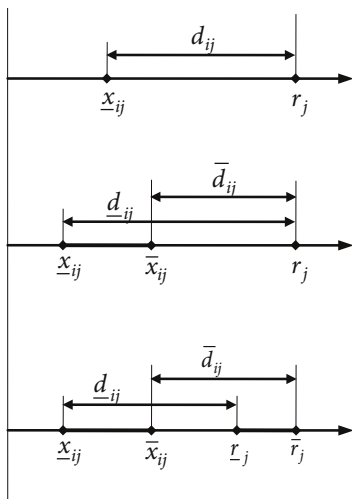

Fig. 2. Characteristic reference point approaches

In proposed approach instead the use of classical reference point[4], we suggest the use of "reference grey point", i.e. reference ellipsoid whose coordinates are represented by appropriate interval grey number with bounds determined using the formulas (62) or (63).

Figure 2 shows three characteristic reference point theory approaches. The first approach (a) shows distance between a point (an exact value) to the reference point. Next approach (b) shows the distance between an interval grey number and classical reference point, and the last approach (c) shows distance between interval grey number and the reference grey point.

The approach (b) in our example gives inadequately rank results. But using the proposed approach, which is graphically shown on Figure 2, we get adequate results. For $\lambda = 0.5$ obtained results is the same as in the Case I. And finally, the λ coefficient provides opportunity to decision maker to give more importance to lower or upper bounds.

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[4] When combine crisp and interval grey numbers, crisp numbers are represented with interval grey number with same upper and lower bounds.

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